Lecture 23: Hypercontractivity

Overview

- Today we shall learn about an advanced tool in Fourier Analysis called Hypercontractivity. We shall see the theorem and a few of its applications. However, we shall not see the proof

L_p Norm

- For $p \geqslant 1$ and any function $f \colon \{0,1\}^n \to \mathbb{R}$, we define

$$L_p(f) := \left(\frac{1}{N} \sum_{x \in \{0,1\}^n} \left| f(x) \right|^p \right)^{1/p}$$

- There are two interesting properties of the $L_p(\cdot)$ norm

Lemma (Monotonicity of Norm)

For $1 \leqslant p < q$ and $f \colon \{0,1\}^n \to \mathbb{R}$ we have

$$L_p(f) \leqslant L_q(f)$$

Moreover, equality holds if and only if f is a constant function

L_p Norm II

- Further, we also have the "Contractivity Property." The smoothed version of the function has a smaller norm than the original function.

Lemma (Contractivity)

For $p \geqslant 1$ and $\rho \in [0, 1]$, we have

$$L_p(T_\rho(f)) \leqslant L_p(f)$$

Equality holds if and only if $\rho = 0$ or f is a constant function.

Hypercontractivity

- By the "contractivity property" we know that

$$L_p(T_\rho(f)) \leqslant L_p(f)$$

- By monotonicity of norm, we know that

$$L_p(T_\rho(f)) \leqslant L_q(T_\rho(f)),$$

  where $q > p$
- However, how does $L_p(f)$ compare with $L_q(T_\rho(f))$?
- Answer: It depends.
- **Hypercontractivity.** Even the q-th norm of $T_\rho(f)$ is smaller than the p-th norm of f if $\rho \leqslant \sqrt{\frac{p-1}{q-1}}$.

- Formally, we have the following result

Theorem (Hypercontractivity)

Let $f\colon \{0,1\}^n \to \mathbb{R}$ be an arbitrary function. For $1 \leqslant p < q$ and $\rho \leqslant \sqrt{\frac{p-1}{q-1}}$ we have

$$L_q(T_\rho(f)) \leqslant L_p(f)$$

- **Proof Outline.**
  - Prove the statement for $1 \leqslant p < q = 2$ (The proof of this statement proceeds by induction on n and the base case of $n = 1$ is the toughest case)
  - Reduce the proof of the statement for the case $2 \leqslant p < q$ to the case of $1 \leqslant p < q = 2$ (using Hölder's inequality)
  - Reduce the proof of the statement for the case $1 \leqslant p < 2 < q$ to the two cases above (using the homomorphic property of the noise operator)

Special Case of $q = 2$

- Let us state the hypercontractivity theorem for the special case of $1 \leqslant p < q = 2$
- Suppose $p = 1 + \delta$, where $\delta \in [0, 1)$
- Suppose $\rho = \sqrt{\frac{p-1}{q-1}} = \delta^{1/2}$
- The hypercontractivity theorem states that

$$L_2(T_\rho(f)) \leqslant L_p(f)$$

- Squaring both sides and applying Parseval's identity to the LHS, we get

$$\sum_{S \in \{0,1\}^n} \delta^{|S|} \widehat{f}(S)^2 \leqslant \left(\frac{1}{N} \sum_{x \in \{0,1\}^n} \left| f(x) \right|^{1+\delta} \right)^{2/1+\delta}$$

KKL Lemma I

- Let $f: \{0,1\}^n \to \{+1, 0, -1\}$. Recall that we denoted boolean function by functions with range $\{+1, -1\}$. Suppose we want to denote boolean functions that are only defined on a subset of $\{0,1\}^n$. In this case, we use functions $\{0,1\}^n \to \{+1, 0, -1\}$. Wherever the function is not defined, it evaluates to 0; otherwise, it takes value $\in \{+1, -1\}$.
- The KKL in "KKL Lemma" stands for "Kahn-Kalai-Linial"
- Note that, for a function $f: \{0,1\}^n \to \{+1, 0, -1\}$, we have

$$L_p(f) = \mathbb{P}\left[f(x) \neq 0 \colon x \xleftarrow{\$} \{0,1\}^n\right]^{1/p}$$

KKL Lemma II

- From the hypercontractivity theorem, we directly have the KKL Lemma

Lemma (KKL Lemma)

For any function $f\colon \{0,1\}^n \to \{+1,0,-1\}$ *and* $\delta \in [0,1)$ *we have*

$$\sum_{S\in\{0,1\}^n} \delta^{|S|}\widehat{f}(S)^2 \leqslant \mathbb{P}\left[f(x) \neq 0 \colon x \xleftarrow{\$} \{0,1\}^n\right]^{2/1+\delta}$$

- **Intuition.** Note that the RHS is $\ll \mathbb{P}\left[f(x) \neq 0 \colon x \xleftarrow{\$} \{0,1\}^n\right]$ because $\delta < 1$, i.e., the ratio of the support of f to the size of the entire sample space N.
On the other hand, the LHS is dominated by the Fourier coefficients on S that have a small support.

So, the inequality states that the total mass of the Fourier coefficients on S that have a small support is $\ll$ the ratio of the support of f to the size of the entire sample space N. Effectively, this lemma states that if a boolean function has a small support then most of its mass of the Fourier coefficients is on the S that have a large support.

- In the next lecture, we shall prove a formal result that makes this intuition concrete. We shall show that the uniform distribution on any large subset $A \subseteq \{0,1\}^n$ fools most large linear tests.