Lecture 32: Some Practice with Fourier Analysis

Overview

Today's lecture is primarily based on the material in Section 3 of the survey by Ronald D. Wolf

Two measures of Similarity

- Consider two Boolean functions $f, g\colon \{0,1\}^n \to \{+1,-1\}$
- Suppose $\mathbb{P}\left[f(x) \neq g(x)\right] = \delta$ (where x is drawn uniformly at random from $\{0,1\}^n$). For succinctness, we shall write it as $\mathbb{P}[f \neq g]$.
- Verify that $\langle f, g \rangle = (1 - 2\delta)$. Equivalently,

$$\langle f, g \rangle = 1 - 2 \cdot \mathbb{P}[f \neq g]$$

- Verify that $\|f - g\|_2^2 = 4 \cdot \mathbb{P}[f \neq g]$

Approximating a Boolean Function

- Suppose $f\colon \{0,1\}^n \to \{+1,-1\}$ is a Boolean function
- Let $\mathcal{C} \subseteq \{0,1\}^n$ be a small subset. For example, $\mathcal{C}$ may be the set of all subsets of size $\leqslant d$, a constant.
- Suppose $\sum_{S\in\mathcal{C}} \widehat{f}(S)^2 \geqslant 1-\varepsilon$. Recall that $\sum_S \widehat{f}(S)^2 = 1$ for a Boolean f. This constraint says that the Fourier coefficient $\widehat{f}(S)$, where $S \in \mathcal{C}$, have most of the spectral weight.
- Let us define a new (real-valued) function $h\colon \{0,1\}^n \to \mathbb{R}$ as follows

$$h := \sum_{S\in\mathcal{C}} \widehat{f}(S)\chi_S$$

- Note that h need not be a Boolean function. Instead, consider the Boolean function $\operatorname{sgn} h$, i.e., the sign of the function h
- Our objective is to prove that f and $\operatorname{sgn} h$ disagree with very low probability

- Here is the proof outline. I am leaving the explanation of each step as an exercise.
  Define $D = \left\{x \in \{0,1\}^n \colon f(x) \neq \operatorname{sgn} h(x)\right\}$.

$$
\begin{aligned}
4\mathbb{P}\left[f \neq \operatorname{sgn} h\right] = \|f - \operatorname{sgn} h\|_2^2 &= \frac{1}{N} \cdot \sum_{x \in D} (f - \operatorname{sgn} h)(x)^2 \\
&\leqslant \frac{4}{N} \cdot \sum_{x \in D} (f - h)(x)^2 \\
&\leqslant 4 \cdot \sum_{S} \widehat{(f - h)}(S)^2 \\
&= 4 \cdot \sum_{S} \left(\widehat{f}(S) - \widehat{h}(S)\right)^2 \\
&= 4 \cdot \sum_{S \notin \mathcal{C}} \widehat{f}(S)^2 \\
&\leqslant 4 \cdot \varepsilon.
\end{aligned}
$$

- Therefore, we have $\mathbb{P}\left[f \neq \operatorname{sgn} h\right] \leqslant \varepsilon$

Advantage in Predicting a Boolean Function I

- Suppose $f\colon \{0,1\}^n \to \{+1,-1\}$ is a Boolean function
- Let $p\colon \{0,1\}^n \to [-1,+1]$ be a *sparse polynomial*. That is, there is a small set $\mathcal{C} \subseteq \{0,1\}^n$ such that $\widehat{p}(S) \neq 0 \implies S \in \mathcal{C}$ (Think: What does this mathematical constraint mean in English?)
- Suppose $\langle f, p \rangle \geqslant \varepsilon$
- We will like to claim that there is a character that has *non-trivial advantage* in predicting f
- Here is the proof outline. The explanation of each step is left as exercise.

$$\begin{aligned}
\varepsilon \leqslant \langle f, p \rangle &= \sum_S \widehat{f}(S) \cdot \widehat{p}(S) \\
&= \sum_{S \in \mathcal{C}} \widehat{f}(S) \cdot \widehat{p}(S) \\
&\leqslant \sqrt{\sum_{S \in \mathcal{C}} \widehat{f}(S)^2} \cdot \|p\|_2
\end{aligned}$$

Advantage in Predicting a Boolean Function II

$$\leqslant \sqrt{\sum_{S \in \mathcal{C}} \widehat{f}(S)^2} \cdot 1.$$

- Therefore, there exists $S^* \in \mathcal{C}$ such that

$$\left|\widehat{f}(S^*)\right| \geqslant \frac{\varepsilon}{\sqrt{|\mathcal{C}|}}$$

- Therefore, there is a character χ_{S^*} that has the non-trivial advantage in predicting the function f

Heavy Fourier Coefficients are Few

- Let $f\colon \{0,1\}^n \to \{+1,-1\}$ be a Boolean function
- A heavy Fourier coefficient is one such that $\left|\widehat{f}(S)\right| \geqslant \varepsilon$
- Define the set of all heavy Fourier coefficients

$$\mathcal{C}_\varepsilon = \left\{ S \in \{0,1\}^n \colon \left|\widehat{f}(S)\right| \geqslant \varepsilon \right\}$$

- Prove that $|\mathcal{C}_\varepsilon| \leqslant \frac{1}{\varepsilon^2}$
- I want to emphasize that the upper bound is *independent of n*