Lecture 09: Independent Bounded Differences Inequality

Overview

- Today we shall see a result referred to as the "Independent Bounded Differences Inequality"
- We shall not see the proof of this result today. In the future, when we prove the "Azuma's Inequality," the proof of this theorem shall follow as a corollary
- Today, we shall see how a large class of concentration results follow as a consequence of this concentration inequality. In fact, one such consequence shall look very similar to the "Talagrand Inequality," which we shall study in the next lecture

Independent Bounded Differences Inequality I

- Let $\Omega_1, \ldots, \Omega_n$ be sample spaces
- Define $\Omega := \Omega_1 \times \cdots \times \Omega_n$
- Let $f \colon \Omega \to \mathbb{R}$
- Let $\mathbb{X} = (\mathbb{X}_1, \ldots, \mathbb{X}_n)$ be a random variable over Ω such that each $\mathbb{X}_i$ is independent and $\mathbb{X}_i$ is a random variable over the sample space Ω_i

Definition

A function $f \colon \Omega \to \mathbb{R}$ has *bounded differences if for all* $x, x' \in \Omega$, *there exists* $i \in \{1, \ldots, n\}$ *such that* x *and* x' *differ only at the* i*-th coordinate, then the output of the function* $\left|f(x) - f(x')\right| \leqslant c_i$.

We state the following bound without proof.

Independent Bounded Differences Inequality II

Theorem (Bounded Difference Inequality)

$$\mathbb{P}\left[f(\mathbb{X}) - \mathbb{E}\left[f(\mathbb{X})\right] \geqslant E\right] \leqslant \exp\left(-2E^2 / \sum_{i=1}^{n} c_i^2\right)$$

Applying the same theorem to $-f$, we deduce that

$$\mathbb{P}\left[f(\mathbb{X}) - \mathbb{E}\left[f(\mathbb{X})\right] \leqslant -E\right] \leqslant \exp\left(-2E^2 / \sum_{i=1}^{n} c_i^2\right)$$

Intuitively, if all $c_i = 1$, the random variable $f(\mathbb{X})$ is concentrated around its expected value $\mathbb{E}\left[f(\mathbb{X})\right]$ within a radius of $\sqrt{n}$

Example

- Note that the Chernoff-Hoeffding's bound is a corollary of this theorem
- Let $\mathcal{G}_{n,p}$ be a random graph over n vertices, where each edge is included in the graph independently with probability p. Note that we have m random variables, one indicator variable for each edge in the graph. Note that the chromatic number of graph is a function with bounded difference.
- Several graph properties like the number of connected components
- Longest increasing subsequence
- Max-load in balls-and-bins experiments
- What about the max-load in the power-of-two-choices?

Applicability and Meaningfulness of the Bounds

- Although the theorem is applicable to a problem, the bound that it produces might not be a meaningful bound
- The bound says that the probability mass is concentrated within $\approx \sqrt{n}$ around the expected value $\mathbb{E}\left[f(\mathbb{X})\right]$
- If the expected value $\mathbb{E}\left[f(\mathbb{X})\right]$ is $\omega(\sqrt{n})$ then the theorem gives a meaningful bound
- However, if $\mathbb{E}\left[f(\mathbb{X})\right]$ is $O(\sqrt{n})$ then the theorem does not give a meaningful bound. For example, the longest increasing subsequence, max-load in balls-and-bins experiments

Hamming Distance

Next we shall see a powerful application of the independent bounded difference inequality. First, let us introduce the definition of Hamming Distance

Definition (Hamming Distance)

Let $x, x' \in \Omega := \Omega_1 \times \cdots \times \Omega_n$. We define

$$d_H(x, x') := \left| \left\{ i \colon 1 \leqslant i \leqslant n \text{ and } x_i \neq x'_i \right\} \right|$$

- The Hamming distance of x and x' bounds the number of indices where x and x' differ
- Let $A \subseteq \Omega$ and $d_H(x, A) := \min_{y \in S} d_H(x, y)$.

Definition

The set A_k is defined as follows

$$A_k := \left\{ x \in \Omega \colon d_H(x, A) \leqslant k \right\}$$

Distance from Dense Sets

Lemma

Let $A \subseteq \Omega$. The following bound holds.

$$\mathbb{P}\left[\mathbb{X} \in A\right] \cdot \mathbb{P}\left[d_H(\mathbb{X}, A) \geqslant E\right] \leqslant \exp(-E^2/2n)$$

Intuition

- Suppose $\mathbb{P}\left[\mathbb{X} \in A\right] = 1/2$, then we have

$$\mathbb{P}\left[\mathbb{X} \in A_{E-1}\right] \geqslant 1 - 2\exp(-E^2/2n)$$

  That is, nearly all points lie within $E \approx \sqrt{n}$ distance from the dense set A
- Note that this result holds for <u>all dense sets</u> A

Proof based on the Bounded Difference Inequality I

- Our objective is to prove that

$$\mathbb{P}\left[\mathbb{X} \in A\right] \cdot \mathbb{P}\left[d_H(\mathbb{X}, A) \geqslant E\right] \leqslant \exp(-E^2/2n).$$

Observe that the above inequality is a consequence of the following second inequality:

$$\min\left\{\mathbb{P}\left[\mathbb{X} \in A\right], \mathbb{P}\left[d_H(\mathbb{X}, A) \geqslant E\right]\right\} \leqslant \exp(-E^2/2n).$$

Therefore, we will prove this second inequality instead.

- Note that $d_H(\cdot, A)$ is a bounded difference function with $c_i = 1$, for $i \in \{1, \ldots, n\}$
- Define $\mu = \mathbb{E}\left[d_H(\mathbb{X}, A)\right]$

Proof based on the Bounded Difference Inequality II

- Consider the inequality (using the independent bounded difference inequality for the lower tail)

$$\mathbb{P}\left[\mathbb{X} \in A\right] = \mathbb{P}\left[d_H(\mathbb{X}, A) - \mu \leqslant -\mu\right] \leqslant \exp(-2\mu^2/n).$$

  We will call this the "density bound."

- Now we are ready to prove the "second inequality."
  1. Case 1. Suppose $E \geqslant 2\mu$.

$$\begin{aligned}\min\left\{\mathbb{P}\left[\mathbb{X} \in A\right], \mathbb{P}\left[d_H(\mathbb{X}, A) \geqslant E\right]\right\} &\leqslant \mathbb{P}\left[d_H(\mathbb{X}, A) \geqslant E\right] \\ &= \mathbb{P}\left[d_H(\mathbb{X}, A) - \mu \geqslant (E - \mu)\right] \\ &\leqslant \exp(-2(E-\mu)^2/n) \\ &\quad \text{(By the upper tail bound)} \\ &\leqslant \exp(-E^2/2n). \\ &\qquad \text{(Because } E \geqslant 2\mu\text{)}\end{aligned}$$

Proof based on the Bounded Difference Inequality III

2. Case 2. Suppose $0 \leqslant E < 2\mu$.

$$\min\left\{\mathbb{P}\left[\mathbb{X} \in A\right], \mathbb{P}\left[d_H(\mathbb{X}, A) \geqslant E\right]\right\} \leqslant \mathbb{P}\left[\mathbb{X} \in A\right]$$
$$\leqslant \exp(-2\mu^2/n)$$
(By the "density bound" inequality)
$$\leqslant \exp(-E^2/2n).$$
(Because $0 \leqslant E < 2\mu$)

- Therefore, irrespective of whether $E \geqslant 2\mu$ or $0 \leqslant E < 2\mu$, the following bound holds

$$\min\left\{\mathbb{P}\left[\mathbb{X} \in A\right], \mathbb{P}\left[d_H(\mathbb{X}, A) \geqslant E\right]\right\} \leqslant \exp(-E^2/2n).$$

This completes the proof of our result.

An Application of "Distance from Dense Sets"

(A Slightly weaker-version of) Chernoff-bound

- Consider a uniform distribution over $\Omega = \{0,1\}^n$
- Let A be the set of all binary strings that have at most $n/2$ 1s. The density of this set is $\geqslant 1/2$
- A string x with $d_H(x, A) \geqslant E$ is equivalent to x having $(n/2) + E$ 1s
- So, the probability of an uniformly sampled binary string has $(n/2) + E$ 1s is at most $2\exp(-E^2/2n)$