

Lecture 01: Mathematical Basics (Summations)

What I am Assuming

- I am assuming that you know asymptotic notations. For example, the big-O, little-O notations

- Let us try to write a closed form expression for the following summation

$$S = \sum_{i=1}^n 1$$

- It is trivial to see that $S = n$

- Now, let us try to write a closed form expression for the following summation

$$S = \sum_{i=1}^n i$$

- We can prove that $S = \frac{n(n+1)}{2}$
 - How do you prove this statement? (Use Induction? Use the formula for the Sum of an Arithmetic Progression?)
- Using Asymptotic Notation, we can say that $S = \frac{n^2}{2} + o(n^2)$

Summation III

- Now, let us try to write a closed form expression for the following summation

$$S = \sum_{i=1}^n i^2$$

- We can prove that $S = \frac{n(n+1)(2n+1)}{6}$
 - Why is the expression on the right an integer? (Prove by induction that 6 divides $n(n+1)(2n+1)$ for all positive integer n)
 - How do you prove this statement? (Use Induction?)
- Using Asymptotic Notation, we can say that $S = \frac{n^3}{3} + o(n^3)$

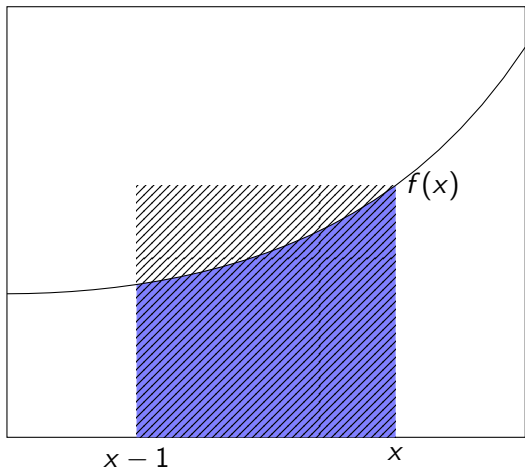
Summation IV

- Do we see a pattern here?
- Conjecture: For $k \geq 1$, we have $\sum_{i=1}^n i^{k-1} = \frac{n^k}{k} + o(n^k)$.
 - How do we prove this statement?

Estimating Summations by Integration I

- Let f be an increasing function
- For example, $f(x) = x^{k-1}$ is an increasing function for $k > 1$ and $x \geq 0$

Estimating Summations by Integration II



Estimating Summations by Integration III

- Observation: “Blue area under the curve” is smaller than the “Shaded area of the rectangle”
 - Blue area under the curve is:

$$\int_{x-1}^x f(t) dt$$

- Shaded area of the rectangle is:

$$f(x)$$

- So, we have the inequality:

$$\int_{x-1}^x f(t) dt \leq f(x)$$

- Summing both side from $x = 1$ to $x = n$, we get

$$\sum_{x=1}^n \int_{x-1}^x f(t) dt \leq \sum_{x=1}^n f(x)$$

Estimating Summations by Integration IV

- The left-hand side of the inequality is

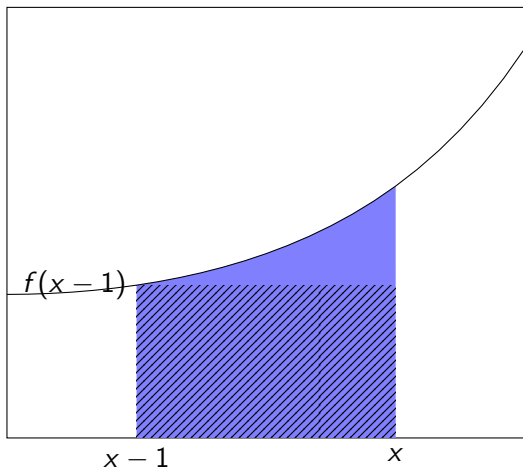
$$\int_0^1 f(t) dt + \int_1^2 f(t) dt + \cdots + \int_{n-1}^n f(t) dt = \int_0^n f(t) dt$$

- So, for an increasing f , we have the following lower bound.

$$\int_0^n f(t) dt \leq \sum_{x=1}^n f(x) \quad (1)$$

Estimating Summations by Integration V

- Now, we will upper bound the summation expression. Consider the figure below



Estimating Summations by Integration VI

- Observation: “Blue area under the curve” is greater than the “Shaded area of the rectangle”
- So, we have the inequality:

$$\int_{x-1}^x f(t) dt \geq f(x-1)$$

- Now we sum the above inequality from $x = 2$ to $x = n + 1$
- We get

$$\int_1^2 f(t) dt + \int_2^3 f(t) dt + \cdots + \int_n^{n+1} f(t) dt \geq f(1) + f(2) + \cdots + f(n)$$

- So, for an increasing f , we get the following upper bound

$$\int_1^{n+1} f(t) dt \geq \sum_{x=1}^n f(x) \quad (2)$$

Summary: Estimation of Summation using Integration

Theorem

For an increasing function f , we have

$$\int_0^n f(t) dt \leq \sum_{x=1}^n f(x) \leq \int_1^{n+1} f(t) dt$$

Exercise:

- Use this theorem to prove that $\sum_{i=1}^n i^{k-1} = \frac{n^k}{k} + o(n^k)$, for $k \geq 1$
- Consider the function $f(x) = 1/x$ to find upper and lower bounds for the sum $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$ using the approach used to prove Theorem 1

Differentiation and Integration

- Differentiation: $f'(x)$ represents the slope of the curve $y = f(x)$ at x
- Integration: $\int_a^b f(t) dt$ represents the area under the curve $y = f(x)$ between $x = a$ and $x = b$
- Increasing function:
 - Observation: The slope an increasing function is positive
 - So, “ f is increasing at x ” is equivalent to “ $f'(x) > 0$,” i.e. f' is positive at x
- Suppose we want to mathematically write “Slope of a function f is increasing”
 - The “slope of a function f ” is the function “ f' ”
 - So, the statement “slope of a function f is increasing” is equivalent to “ $(f')' \equiv f''$ is positive”

Concave Upwards Functions

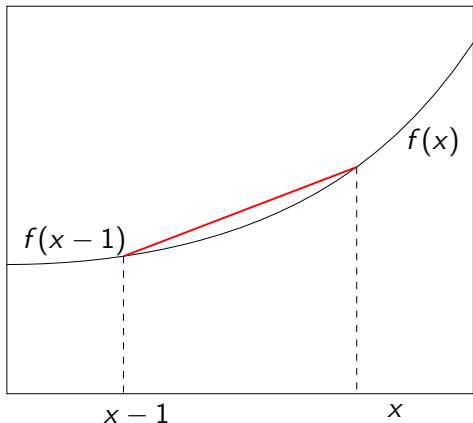
Definition (Concave Upwards Function)

A function f is *concave upwards* in the interval $[a, b]$ if f'' is positive in the interval $[a, b]$.

- Example of functions that concave upwards: x^2 , $\exp(x)$, $1/x$ (in the interval $(0, \infty)$), $x \log x$ (in the interval $(0, \infty)$)
 - We emphasize that a “concave upwards” function need not be increasing, for example $f(x) = 1/x$ (for positive x) is decreasing

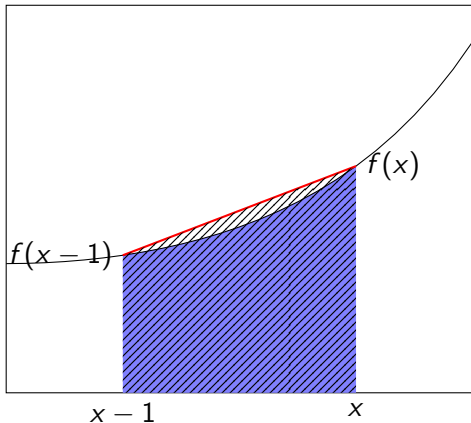
Property of Concave Upwards Function I

- Consider the coordinates $(x - 1, f(x - 1))$ and $(x, f(x))$
- For a concave upwards function, the secant between the two coordinates is always (on or) above the part of the curve f between the two coordinates



Property of Concave Upwards Function II

- So, the shaded area of the trapezium is greater than the blue area under the curve



Property of Concave Upwards Function III

- So, we get

$$\frac{f(x-1) + f(x)}{2} \geq \int_{x-1}^x f(t) dt$$

- Now, use this new observation to obtain a better lower bound for the sum $\sum_{x=1}^n f(x)$
- Think: Can you get even tighter bounds?
- Additional Reading: Read on the “trapezoidal rule”

Estimating Products

- Consider the objective of estimating $n!$ using elementary functions
- Note that one can convert this estimation of products into estimation of sums by taking log. For example,

$$\ln(n!) = \sum_{i=1}^n \ln(i).$$

- Now, one can tightly upper and lower bound the expression $\sum_{i=1}^n \ln(i)$. Use the techniques in the previous slides to obtain meaningful upper and lower bounds of this expression. Suppose

$$L_n \leq \sum_{i=1}^n \ln(i) \leq U_n.$$

- Therefore, one concludes that

$$\exp(L_n) \leq n! \leq \exp(U_n).$$

Estimating Fractions

- Consider the objective of estimating a fraction A_n/B_n
- Suppose we have $A_n \leq U_n$ and $L'_n \leq B_n$. Note that

$$\frac{1}{B_n} \leq \frac{1}{L'_n}.$$

- Note that multiplying with $A_n \leq U_n$, one gets that

$$\frac{A_n}{B_n} \leq \frac{U_n}{L'_n}.$$

- To summarize, upper-bounding a fraction involves upper-bounding the numerator and lower-bounding the denominator
- Analogously, if $L_n \leq A_n$ and $B_n \leq U'_n$, then we get $\frac{L_n}{B_n} \leq \frac{A_n}{U'_n}$
- **Food for thought.** Provide meaningful upper and lower bound the expression $\binom{2n}{n} := \frac{(2n)!}{(n!)^2}$.