Homework 2

1. **Solving an Interesting Equation.** (20 points) Our objective is to understand the behavior of x such that $x! = n$ as a function of n. We shall use the following estimate of $x!$

$$\left(\frac{x}{\mathrm{e}}\right)^x \leqslant x! \leqslant x^x$$

(Remark: The upper-bound is trivial. The lower-bound is a consequence of estimating the increasing function $\log t$ using integrals.)

- Prove that if $x = \frac{\log n}{\log\log n}$ then $x! \leqslant n$.
- Prove that, for large enough n, if $x = \frac{\mathrm{e}\log n}{\log\log n}$ then $x! \geqslant n$.

(Remark: These proofs complete the argument that $x = \Theta(\log n/\log\log n)$. Substituting $\mathrm{poly}\, n$ instead of n, completes the argument that $x = \Theta(\log n/\log\log n)$ when $x! = \mathrm{poly}\, n$, for any fixed polynomial poly)

Solution.

2. **Upper-bounding Max-load using Poisson Approximation Theorem.** (20 points) Recall that in the lecture we proved the upper-bound on max-load directly. Let us see how we can use the Poisson approximation theorem to prove that result easily.

   - Let $\mathbb{X}(\mu)$ be the Poisson distribution with mean μ. Prove the following bound. For any integer $T \geqslant 2\mu$, we have

     $$\mathbb{P}\left[\mathbb{X}(\mu) \geqslant T\right] \leqslant 2\mathbb{P}\left[\mathbb{X}(\mu) = T\right]$$

     (Remark: Basically, this inequality proves that $\mathbb{P}\left[\mathbb{X}(\mu) \geqslant T\right]$ is well approximated by $\mathbb{P}\left[\mathbb{X}(\mu) = T\right]$)
   - Suppose $\mathbb{X}$ represents the Poisson distribution with mean $\mu = 1$. Prove that there exists a positive constant c such that

     $$\mathbb{P}\left[\mathbb{X} \geqslant c\frac{\log n}{\log\log n}\right] \leqslant \frac{1}{n^3}$$
   - Prove that

     $$\mathbb{P}\left[\max\left\{\mathbb{X}^{(1)}, \mathbb{X}^{(2)}, \dots, \mathbb{X}^{(n)}\right\} < c\frac{\log n}{\log\log n}\right] \geqslant 1 - \frac{1}{n^2}$$

Solution.

3. **Coupon Collector Problem.** (20 points) Our objective is to solve the Coupon Collector Problem using the Poisson approximation theorem. Here, we want to determine the value of m such that when m balls are thrown into n bins, with high probability every bin receives at least one ball. Equivalently, we want to determine the value of m such that the probability of the minimum load being 0 is small.

   - Let $\mathbb{X}(\mu)$ be the Poisson distribution with mean μ. Find the value of m such that, for $\mu = m/n$, we have
   $$\mathbb{P}\left[\mathbb{X}(\mu) = 0\right] \leqslant \frac{1}{n^3}$$
   - Let $\mathbb{X}$ be the Poisson distribution for the μ determined above. Let $\mathbb{X}^{(i)}$ be the i-th independent copy of the distribution $\mathbb{X}$. Prove the following bound.
   $$\mathbb{P}\left[\min\left\{\mathbb{X}^{(1)}, \mathbb{X}^{(2)}, \dots, \mathbb{X}^{(n)}\right\} = 0\right] \leqslant \frac{1}{n^2}$$
   - Use the Poisson Approximation Theorem to prove the following bound. For large enough n and $\mathbb{L}_{\min} := \min\left\{\mathbb{L}_1, \mathbb{L}_2, \dots, \mathbb{L}_n\right\}$
   $$\mathbb{P}\left[\mathbb{L}_{\min} \geqslant 1\right] \geqslant 1 - \frac{1}{n}$$

   Solution.

4. **A Fun Ball and Bins Problem.** (20 points) Let us consider a fun problem. Suppose we are interested in ensuring that every bin received at least 2 balls. Let us get you started on how to think about this fun problem.

   Let $\mathbb{X}(\mu)$ be the Poisson distribution with mean μ. Find the value of m such that, for $\mu = m/n$ and positive integers m and n, we have

$$\mathbb{P}\left[\mathbb{X}(\mu) \in \{0,1\}\right] \leqslant \frac{1}{n^3}$$

   Solution.

5. **Towards proving Poisson Approximation Theorem.** (20 points) Let me get you started towards proving the simpler version of the Poisson approximation theorem that was taught in the class. Let $\mathbb{X}$ be the Poisson distribution with mean μ, where $\mu = m/n$. Prove the following inequality

$$\mathbb{P}\left[\mathbb{X}^{(1)} + \mathbb{X}^{(2)} + \cdots + \mathbb{X}^{(n)} = m\right] = \left(\frac{m}{\mathrm{e}}\right)^m \frac{1}{m!} \geqslant \frac{1}{\mathrm{e}\sqrt{m}}$$

(Remark: Use the Stirling's approximation taught in the class for the final inequality.)

Solution.

Collaborators :