Lecture 30: Discrete Fourier Analysis on the Boolean Hypercube (Introduction)
Recall

- Our objective is to study a function $f : \{0, 1\}^n \to \mathbb{R}$
- Every function $f$ is equivalently represented as the vector
  $$(f(0), f(1), \ldots, f(N - 1)) \in \mathbb{R}^N,$$
  where $N = 2^n$
- For $S = S_1S_2\ldots S_n \in \{0, 1\}^n$, define the following function
  $$\chi_S(x) := (-1)^{S_1x_1+S_2x_2+\cdots+S_nx_n},$$
  where $x = x_1x_2\ldots x_n \in \{0, 1\}^n$
- We defined an inner-product of functions
  $$\langle f, g \rangle := \frac{1}{N} \sum_{x \in \{0,1\}^n} f(x)g(x)$$
  We showed that $\left\{ \chi_S : S \in \{0, 1\}^N \right\}$ is an orthonormal basis.
  That is,
  $$\langle \chi_S, \chi_T \rangle = \begin{cases} 
  0, & \text{if } S \neq T \\
  1, & \text{if } S = T 
  \end{cases}$$
Fourier Coefficients

Since \( \chi_S : S \in \{0, 1\}^n \) is an orthonormal basis, we can express any \( f \) as follows

\[
f = \hat{f}(0)\chi_0 + \hat{f}(1)\chi_1 + \cdots + \hat{f}(N - 1)\chi_{N-1},
\]

where \( \hat{f}(S) \in \mathbb{R} \) and \( S \in \{0, 1\}^n \).

- We interpret \( (\hat{f}(0), \hat{f}(1), \ldots, \hat{f}(N - 1)) \) as a function \( \hat{f} \)
Fourier Transformation is a basis change that maps the function \( f \) to the function \( \hat{f} \).

We shall represent it as \( f \mapsto \hat{f} \), where \( \mathcal{F} \) is the Fourier Transformation.
Linearity of Fourier Transformation

- Note that we have the following property. For any $S \in \{0, 1\}^n$, we have $\langle f, \chi_S \rangle = \hat{f}(S)$. So, we get

$$ (f(0)f(1) \cdots f(N - 1)) \cdot \frac{1}{N} \chi_S(0)\chi_S(1) \cdots \chi_S(N - 1) = \hat{f}(S) $$

- Define the matrix

$$ F = \frac{1}{N} \begin{bmatrix}
\chi_0(0) & \chi_1(0) & \cdots & \chi_{N-1}(0) \\
\chi_0(1) & \chi_1(1) & \cdots & \chi_{N-1}(1) \\
\vdots & \vdots & \ddots & \vdots \\
\chi_0(N - 1) & \chi_1(N - 1) & \cdots & \chi_{N-1}(N - 1)
\end{bmatrix} $$

- From the property mentioned above, note that we have the identity

$$ f \cdot F = \hat{f} $$
Claim

For two functions $f, g: \{0, 1\}^n \to \mathbb{R}$, we have

\[
\hat{(f + g)} = \hat{f} + \hat{g}
\]

Proof.

\[
\hat{(f + g)} = (f + g)\mathcal{F} = f\mathcal{F} + g\mathcal{F} = \hat{f} + \hat{g}
\]
Claim

For a function $f : \{0, 1\}^n \to \mathbb{R}$ and $c \in \mathbb{R}$, we have

$$\hat{(cf)} = c \hat{f}$$

Proof.

$$\hat{(cf)} = (cf) \mathcal{F} = c(f \mathcal{F}) = c \hat{f}$$
Theorem

Let $f : \{0, 1\}^n \rightarrow \mathbb{R}$. Then, we have

$$\widehat{\langle f \rangle} = \frac{1}{N} \cdot f$$

Proof.

- We shall prove that $\mathcal{F} \cdot \mathcal{F} = \frac{1}{N} I_{N \times N}$. This result shall directly imply that $\widehat{(f \mathcal{F})} = \frac{1}{N} \cdot f$

- Let us compute the element $(\mathcal{F} \cdot \mathcal{F})_{i,j}$. This element is the product of the $i$-th row of $\mathcal{F}$ and the $j$-th column of $\mathcal{F}$

- The $j$-th column of $\mathcal{F}$ is $\left(\frac{1}{N} \chi_j\right)^\top$

- The $i$-th row of $\mathcal{F}$ is $\left(\chi_0(i) \chi_1(i) \cdots \chi_{N-1}(i)\right)$

- Note that $\chi_S(x) = \chi_x(S)$, i.e., the matrix $\mathcal{F}$ is symmetric
So, the $i$-th row of $\mathcal{F}$ is $\frac{1}{N}\chi_i$

Therefore, we have $(\mathcal{F}\mathcal{F})_{i,j} = \frac{1}{N^2} \cdot \chi_i \cdot \chi_j^\top = \frac{1}{N} \langle \chi_i, \chi_j \rangle$. The orthonormality of the Fourier basis completes the proof.
Theorem (Plancherel) 

Suppose $f, g : \{0, 1\}^n \to \mathbb{R}$. Then, the following holds 

$$
\langle f, g \rangle = \sum_{S \in \{0, 1\}^n} \hat{f}(S) \hat{g}(S)
$$
Proof.

\[
\langle f, g \rangle = \left\langle \sum_{S \in \{0,1\}^n} \hat{f}(S) \chi_S, \sum_{T \in \{0,1\}^n} \hat{g}(T) \chi_T \right\rangle \\
= \sum_{S \in \{0,1\}^n} \hat{f}(S) \left\langle \chi_S, \sum_{T \in \{0,1\}^n} \hat{g}(T) \right\rangle \\
= \sum_{S \in \{0,1\}^n} \hat{f}(S) \sum_{T \in \{0,1\}^n} \hat{g}(T) \langle \chi_S, \chi_T \rangle \\
= \sum_{S \in \{0,1\}^n} \hat{f}(S) \hat{g}(S)
\]
Note that, if \( f, g : \{0, 1\}^n \rightarrow \{+1, -1\} \) and we have \( \langle f, g \rangle = 1 - \varepsilon \), then \( f \) and \( g \) disagree at \( \varepsilon N \) inputs. Intuitively, if \( |\langle f, g \rangle| \) is close to 1 then the functions are highly correlated. On the other hand, if \( |\langle f, g \rangle| \) is close to 0 then the functions are independent.
Theorem (Parseval’s Identity)

Suppose \( f : \{0, 1\}^n \to \mathbb{R} \). Then

\[
\langle f, f \rangle = \sum_{S \in \{0, 1\}^n} \hat{f}(S)^2
\]

Substitute \( f = g \) in Plancherel’s theorem.
Corollary

If $f : \{0, 1\}^n \to \{+1, -1\}$, then $\sum_{S \in \{0, 1\}^n} \hat{f}(S)^2 = 1$

Follows from the fact that $\langle f, f \rangle = 1$ and the Parseval’s identity