Lecture 29: Discrete Fourier Analysis on the Boolean Hypercube (Introduction)

Focus of Study

- Functions with domain $\{0,1\}^n$ and range $\mathbb{R}$
- Let $f\colon \{0,1\}^n \to \mathbb{R}$
- We shall always use $N = 2^n$
- Any n-bit binary string shall be canonically interpreted as an integer in the range $\{0,1,\dots,N-1\}$
- For any function $f\colon \{0,1\}^n \to \mathbb{R}$ we shall associate the following unique vector in $\mathbb{R}^N$

$$\big(f(0), f(1), \dots, f(N-1)\big)$$

Kronecker Basis

- For $i \in \{0, 1, \ldots, N-1\}$, we define the function $\delta_i \colon \{0,1\}^n \to \mathbb{R}$ as follows

$$\delta_i(x) = \begin{cases} 1, & \text{if } x = i \\ 0, & \text{otherwise} \end{cases}$$

- Note that the functions $\{\delta_0, \delta_1, \ldots, \delta_{N-1}\}$ form a basis for $\mathbb{R}^N$
- Any function f can be expressed as a linear combination of these basis functions as follows

$$f = f(0)\delta_0 + f(1)\delta_1 + \cdots + f(N-1)\delta_{N-1}$$

- Our goal is to study the function f in a new basis, namely, the "Fourier Basis," that shall be introduced next. We emphasize that this basis need not be unique

- For $S = (S_1, S_2, \ldots, S_n) \in \{0,1\}^n$, we define the following function
$$\chi_S(x) := (-1)^{\sum_{i=1}^n S_i \cdot x_i}$$
- Several introductory materials on Fourier analysis interpret S as a subset of $\{1, 2, \ldots, n\}$. Although, the definition presented here is equivalent to this interpretation, I personally prefer this notation because it generalized to other domains.

An Example

- Suppose $n = 3$ and we are working with functions $f \colon \{0,1\}^n \to \mathbb{R}$
- Note that there are 8 different Fourier basis functions

$$\chi_{000}(x) = (-1)^0 = 1$$
$$\chi_{100}(x) = (-1)^{x_1}$$
$$\chi_{010}(x) = (-1)^{x_2}$$
$$\chi_{110}(x) = (-1)^{x_1+x_2}$$
$$\chi_{001}(x) = (-1)^{x_3}$$
$$\chi_{101}(x) = (-1)^{x_1+x_3}$$
$$\chi_{011}(x) = (-1)^{x_2+x_3}$$
$$\chi_{111}(x) = (-1)^{x_1+x_2+x_3}$$

(All non-trivial) Basis Functions are balanced I

Lemma

$$\sum_{x\in\{0,1\}^n} \chi_R(x) = \begin{cases} N, & \textit{if } R = 0 \\ 0, & \textit{otherwise} \end{cases}$$

Proof:

- Suppose $R = 0$, then we have

$$\sum_{x\in\{0,1\}^n} \chi_R(x) = \sum_{x\in\{0,1\}^n} 1 = N$$

- Suppose $R \neq 0$. Let $\{i_1, i_2, \ldots, i_r\}$ be the set of indices $\{i \colon R_i = 1\}$

$$
\begin{aligned}
\sum_{x \in \{0,1\}^n} \chi_R(x) &= \sum_{x \in \{0,1\}^n} (-1)^{R_1 x_1 + \cdots + R_n x_n} \\
&= \sum_{x \in \{0,1\}^n} (-1)^{R_{i_1} x_{i_1} + \cdots + R_{i_r} x_{i_r}} \\
&= \sum_{x_{-i_1} \in \{0,1\}^{n-1}} (-1)^{R_{i_2} x_{i_2} + \cdots + R_{i_r} x_{i_r}} \sum_{x_{i_1} \in \{0,1\}} (-1)^{x_{i_1}} \\
&= \sum_{x_{-i_1} \in \{0,1\}^{n-1}} (-1)^{R_{i_2} x_{i_2} + \cdots + R_{i_r} x_{i_r}} \left((-1)^0 + (-1)^1 \right) \\
&= \sum_{x_{-i_1} \in \{0,1\}^{n-1}} (-1)^{R_{i_2} x_{i_2} + \cdots + R_{i_r} x_{i_r}} \cdot 0 = 0
\end{aligned}
$$

Inner Product

Definition (Inner Product)

The inner-product of two functions $f, g \colon \{0,1\}^n \to \mathbb{R}$ is defined as follows

$$\langle f, g \rangle := \frac{1}{N} \sum_{x \in \{0,1\}^n} f(x) g(x)$$

Orthonormality of the Basis Functions I

Lemma

$$\langle \chi_S, \chi_T \rangle = \begin{cases} 1, & \text{if } S = T \\ 0, & \text{otherwise} \end{cases}$$

Proof:

- $$\langle \chi_S, \chi_T \rangle = \frac{1}{N} \sum_{x \in \{0,1\}^n} \chi_S(x) \chi_T(x)$$
  $$= \frac{1}{N} \sum_{x \in \{0,1\}^n} (-1)^{(S_1+T+1)x_1 + \ldots + (S_n+T_n)x_n}$$
- Note that if $S_i = T_i$ then $(-1)^{(S_i+T_i)x_i} = 1$; otherwise $(-1)^{(S_i+T_i)x_i} = (-1)^{x_i}$

Orthonormality of the Basis Functions II

- Define R such that $R_i = 1$ if $S_i \neq T_i$; otherwise $R_i = 0$
- Then, the right-hand side expression becomes

$$\begin{aligned}
\langle \chi_S, \chi_T \rangle &= \frac{1}{N} \sum_{x \in \{0,1\}^n} (-1)^{R_1 x_1 + \cdots + R_n x_n} \\
&= \frac{1}{N} \sum_{x \in \{0,1\}^n} \chi_R(x) \\
&= \begin{cases} \frac{1}{N} \cdot N, & \text{if } R = 0 \\ \frac{1}{N} \cdot 0, & \text{otherwise} \end{cases}
\end{aligned}$$

- Note that $R = 0$ if and only if $S = T$. This observation completes the proof