Lecture 29: Discrete Fourier Analysis on the Boolean Hypercube (Introduction)
Focus of Study

- Functions with domain $\{0, 1\}^n$ and range $\mathbb{R}$
- Let $f : \{0, 1\}^n \rightarrow \mathbb{R}$
- We shall always use $N = 2^n$
- Any $n$-bit binary string shall be canonically interpreted as an integer in the range $\{0, 1, \ldots, N - 1\}$
- For any function $f : \{0, 1\}^n \rightarrow \mathbb{R}$ we shall associate the following unique vector in $\mathbb{R}^N$

$$ (f(0), f(1), \ldots, f(N - 1)) $$
For $i \in \{0, 1, \ldots, N - 1\}$, we define the function
\[ \delta_i : \{0, 1\}^n \rightarrow \mathbb{R} \text{ as follows} \]
\[ \delta_i(x) = \begin{cases} 1, & \text{if } x = i \\ 0, & \text{otherwise} \end{cases} \]

Note that the functions \{\delta_0, \delta_1, \ldots, \delta_{N-1}\} form a basis for \( \mathbb{R}^N \)

Any function \( f \) can be expressed as a linear combination of
these basis functions as follows
\[ f = f(0)\delta_0 + f(1)\delta_1 + \cdots + f(N - 1)\delta_{N-1} \]

Our goal is to study the function \( f \) in a new basis, namely, the
“Fourier Basis,” that shall be introduced next. We emphasize
that this basis need not be unique.
Fourier Basis Functions

For $S = (S_1, S_2, \ldots, S_n) \in \{0, 1\}^n$, we define the following function

$$
\chi_S(x) := (-1)^{\sum_{i=1}^n S_i \cdot x_i}
$$

Several introductory materials on Fourier analysis interpret $S$ as a subset of $\{1, 2, \ldots, n\}$. Although, the definition presented here is equivalent to this interpretation, I personally prefer this notation because it generalized to other domains.
Suppose $n = 3$ and we are working with functions $f : \{0, 1\}^n \to \mathbb{R}$.

Note that there are 8 different Fourier basis functions:

- $\chi_{000}(x) = (-1)^0 = 1$
- $\chi_{100}(x) = (-1)^{x_1}$
- $\chi_{010}(x) = (-1)^{x_2}$
- $\chi_{110}(x) = (-1)^{x_1 + x_2}$
- $\chi_{001}(x) = (-1)^{x_3}$
- $\chi_{101}(x) = (-1)^{x_1 + x_3}$
- $\chi_{011}(x) = (-1)^{x_2 + x_3}$
- $\chi_{111}(x) = (-1)^{x_1 + x_2 + x_3}$
Lemma

\[ \sum_{x \in \{0,1\}^n} \chi_R(x) = \begin{cases} N, & \text{if } R = 0 \\ 0, & \text{otherwise} \end{cases} \]

Proof:

- Suppose \( R = 0 \), then we have

\[ \sum_{x \in \{0,1\}^n} \chi_R(x) = \sum_{x \in \{0,1\}^n} 1 = N \]
Suppose $R \neq 0$. Let $\{i_1, i_2, \ldots, i_r\}$ be the set of indices $\{i: R_i = 1\}$

$$\sum_{x \in \{0,1\}^n} \chi_R(x) = \sum_{x \in \{0,1\}^n} (-1)^{R_1 x_1 + \cdots + R_n x_n}$$

$$= \sum_{x \in \{0,1\}^n} (-1)^{R_{i_1} x_{i_1} + \cdots + R_{i_r} x_{i_r}}$$

$$= \sum_{x_{-i_1} \in \{0,1\}^{n-1}} (-1)^{R_{i_2} x_{i_2} + \cdots + R_{i_r} x_{i_r}} \sum_{x_{i_1} \in \{0,1\}} (-1)^{x_{i_1}}$$

$$= \sum_{x_{-i_1} \in \{0,1\}^{n-1}} (-1)^{R_{i_2} x_{i_2} + \cdots + R_{i_r} x_{i_r}} \left((-1)^0 + (-1)^1\right)$$

$$= \sum_{x_{-i_1} \in \{0,1\}^{n-1}} (-1)^{R_{i_2} x_{i_2} + \cdots + R_{i_r} x_{i_r}} \cdot 0 = 0$$
Inner Product

Definition (Inner Product)

The inner-product of two functions \( f, g : \{0, 1\}^n \rightarrow \mathbb{R} \) is defined as follows

\[
\langle f, g \rangle := \frac{1}{N} \sum_{x \in \{0, 1\}^n} f(x)g(x)
\]
Lemma

\[ \langle \chi_S, \chi_T \rangle = \begin{cases} 1, & \text{if } S = T \\ 0, & \text{otherwise} \end{cases} \]

Proof:

\[ \langle \chi_S, \chi_T \rangle = \frac{1}{N} \sum_{x \in \{0,1\}^n} \chi_S(x) \chi_T(x) \]

\[ = \frac{1}{N} \sum_{x \in \{0,1\}^n} (-1)^{(S_1+T_1)x_1 + \ldots + (S_n+T_n)x_n} \]

- Note that if \( S_i = T_i \) then \( (-1)^{(S_i+T_i)x_i} = 1 \); otherwise \( (-1)^{(S_i+T_i)x_i} = (-1)^{x_i} \)
Define $R$ such that $R_i = 1$ if $S_i \neq T_i$; otherwise $R_i = 0$

Then, the right-hand side expression becomes

$$\langle \chi_S, \chi_T \rangle = \frac{1}{N} \sum_{x \in \{0,1\}^n} (-1)^{R_1x_1 + \cdots + R_nx_n}$$

$$= \frac{1}{N} \sum_{x \in \{0,1\}^n} \chi_R(x)$$

$$= \begin{cases} \frac{1}{N} \cdot N, & \text{if } R = 0 \\ \frac{1}{N} \cdot 0, & \text{otherwise} \end{cases}$$

Note that $R = 0$ if and only if $S = T$. This observation completes the proof