Lecture 25: Generalized Lovász Local Lemma

Recall: Lovász Local Lemma

- We design an experiment with independent random variables $\mathbb{X}_1, \ldots, \mathbb{X}_m$
- We define bad events $\mathbb{B}_1, \ldots, \mathbb{B}_n$, where the bad event $\mathbb{B}_i$ depends on the variables $\mathbb{X}_{k_1}, \ldots, \mathbb{X}_{k_{n_i}}$
- We defien $\mathsf{Vbl}_i := \{k_1, \ldots, k_{n_i}\}$, the set of all variables that the bad event $\mathbb{B}_i$ depends on
- The bad event $\mathbb{B}_i$ can depend on the bad event $\mathbb{B}_j$ if $\mathsf{Vbl}_i \cap \mathsf{Vbl}_j \neq \emptyset$
- Suppose each bad event $\mathbb{B}_i$ depends on at most d other bad events
- Suppose we show that, for each bad event $\mathbb{B}_i$, the probability of its occurrence $\mathbb{P}\left[\mathbb{B}_i\right] \leqslant p$
- If $\mathrm{e}p(d+1) \leqslant 1$, then

$$\mathbb{P}\left[\overline{B_1}, \ldots, \overline{B_n}\right] \geqslant \left(1 - \frac{1}{d+1}\right)^n > 0$$

Generalized Lovász Local Lemma

- We design an experiment with independent random variables $\mathbb{X}_1, \ldots, \mathbb{X}_m$
- We define bad events $\mathbb{B}_1, \ldots, \mathbb{B}_n$
- Let D_i be the set of indices of other bad events that $\mathbb{B}_i$ depends on
- Suppose we exhibit the existence of numbers $(x_1, \ldots, x_n)$ such that the following holds. For each $i \in \{1, \ldots, n\}$, we have

$$\mathbb{P}\left[\mathbb{B}_i\right] \leqslant x_i \prod_{j \in D_i} (1 - x_j)$$

- Then, the following holds

$$\mathbb{P}\left[\overline{\mathbb{B}_1}, \ldots, \overline{\mathbb{B}_n}\right] \geqslant \prod_{i=1}^{n} (1 - x_i) > 0$$

Notes

- Prove Lovász Local Lemma using the Generalized Lovász Local Lemma
- The numbers $(x_1, \ldots, x_n)$ are not probabilities that add up to 1. This intuition is incorrect
- Prove the following corollary of the generalized Lovász Local Lemma

Corollary

If for all $i \in \{1, \ldots, n\}$, we have $\sum_{j \in D_i} \mathbb{P}\left[\mathbb{B}_j\right] < 1/4$, then

$$\mathbb{P}\left[\overline{\mathbb{B}_1}, \ldots, \overline{\mathbb{B}_n}\right] \geqslant \prod_{i=1}^{n}\left(1 - 2\mathbb{P}\left[\mathbb{B}_j\right]\right) > 0$$

- Prove the results in the previous lecture using this corollary albeit with a slight worse parameter choices

Frugal Coloring of a Graph

Definition

A β-*frugal coloring* of a graph G satisfies the following two conditions.

1. It is a valid coloring, and
2. In the neighborhood $N_G(v)$ of any vertex $v \in V(G)$, there are at most β vertices with the same color

For example, a 1-frugal coloring of a graph is a coloring of the graph G^2

Example: Frugal Coloring

We shall show the following result

Theorem

For $\beta \in \mathbb{N}$ and a graph G with maximum degree $\Delta \geqslant \beta^\beta$ there exists a β-frugal coloring using $16\Delta^{1+1/\beta}$ colors.

Note that a graph with maximum degree Δ can be 1-frugally colored using $\Delta^2 + 1$ colors. The theorem mentioned above uses asymptotically the same number of colors. We shall prove the general result using the corollary of the generalized Lovász Local Lemma

Randomly color the vertices of the graph with C colors. We shall consider two types of bad events.

- $\mathbb{B}_e$, where $e \in E(G)$. If the two vertices at the endpoints of the edge e receive the same color then this bad event occurs. There will be type-1 bad events.
- $\mathbb{B}_{u_1,\dots,u_{\beta+1}}$, where $u_1, \dots, u_{\beta+1} \in V(G)$. Suppose there exists a vertex v such that $u_1, \dots, u_{\beta+1}$ are distinct vertices in $N_G(v)$ with identical colors. These will be called type-2 bad events.

Example: Frugal Coloring III

- Note that one type-1 bad event $\mathbb{B}_e$ can depend on at most 2Δ other type-1 bad events $\mathbb{B}_{e'}$
- We are now interested in computing the number of type-2 bad events that $\mathbb{B}_e$ can depend on. Consider a type-2 bad event $\mathbb{B}_{u_1,\ldots,u_{\beta+1}}$ such that there exists $v \in V(G)$ such that $u_1, \ldots, u_{\beta+1} \in N_G(v)$. Suppose that the edge $e = (a, b)$. Note that a has at most Δ neighbors. So, there are at most Δ possible ways of choosing v. Note that we have $\binom{\Delta}{\beta}$ ways of choosing the remaining vertices $\{u_1, \ldots, u_{\beta+1}\} \setminus \{a\}$. Similarly, the case of the vertex b as well. So, there are at most $2\Delta\binom{\Delta}{\beta}$ type-2 events that $\mathbb{B}_e$ can depend on

- Similarly, a type-2 event $\mathbb{B}_{u_1,\dots,u_{\beta+1}}$ can depend on at most $(\beta+1)\Delta$ other type-1 bad events and $(\beta+1)\Delta\binom{\Delta}{\beta}$ other type-2 bad events
- Note that

$$\mathbb{P}\left[\mathbb{B}_e\right] \leqslant \frac{1}{C}$$
$$\mathbb{P}\left[\mathbb{B}_{u_1,\dots,u_{\beta+1}}\right] \leqslant \frac{1}{C^\beta}$$

- So, to prove that a β-frugal coloring exists using the corollary of the generalized Lovász Local Lemma, it suffices to prove that

$$(\beta+1)\Delta\cdot\frac{1}{C} + (\beta+1)\Delta\binom{\Delta}{\beta}\cdot\frac{1}{C^\beta} < \frac{1}{4}$$

Example: Frugal Coloring V

- We can use the upper-bound $\binom{n}{k} \leqslant \left(\frac{en}{k}\right)^k$ to upper-bound the expression

$$(\beta+1)\Delta \cdot \frac{1}{C} + (\beta+1)\Delta\binom{\Delta}{\beta} \cdot \frac{1}{C^\beta}$$

- This is left as an exercise