Lecture 16: Talagrand Inequality Application

Longest Increasing Subsequence I

- Suppose $\mathbb{X} = (\mathbb{X}_1, \ldots, \mathbb{X}_n)$, where each $\mathbb{X}_i$ is independent and uniformly distributed over $\Omega_i = [0, 1)$
- We are interested in demonstrating a concentration bound for $f(\mathbb{X})$, where $f(\mathbb{X})$ is the longest increasing subsequence in $(\mathbb{X}_1, \ldots, \mathbb{X}_n)$
- **Observation.** Consider any $x \in \Omega := \Omega_1 \times \cdots \times \Omega_n$. If $f(x) = k$ (i.e., the longest increased subsequence in x is k), then there is a set $K_x = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ such that K_x denotes the indices of the longest increasing subsequence in x
- **Observation.** Consider any $y \in \Omega$. Note that if y agrees with x at all the indices in K_x, then we have $f(y) \geqslant f(x)$ (it is possible that y has a longest increasing subsequence, but, definitely, it will not be shorter than the length of the longest increasing subsequence in x)

- **Observation.** Let us generalize the previous observation further. Consider any $y \in \Omega$. Note that if y agrees with x at all indices in K_x except at ℓ indices. Then, we have $f(y) \geqslant f(x) - \ell$. Formally, we can write this as follows

$$f(y) \geqslant f(x) - \left|\{i \colon i \in K_x \text{ and } x_i \neq y_i\}\right|$$

- Intuitively, we incur a penalty for every $i \in K_x$ where x and y differ. Let us fix $\alpha_x = (\alpha_1, \ldots, \alpha_n)$ such that

$$\alpha_i = \begin{cases} 0 & i \notin K_x \\ \frac{1}{\sqrt{|K_x|}} & i \in K_x \end{cases}$$

Note that $|K_x| = f(x)$. So, we conclude that

$$f(y) \geqslant f(x) - \sqrt{f(x)}\, d_{\alpha_x}(x, y)$$

Longest Increasing Subsequence III

- Rearranging, we get that

$$d_{\alpha_x}(x, y) \geqslant \frac{f(x) - f(y)}{\sqrt{f(x)}}$$

- Since, $d_T(\cdot, \cdot)$ is a supremum of $d_\alpha(\cdot, \cdot)$ over all α with norm-1, we get that

$$d_T(x, y) \geqslant \frac{f(x) - f(y)}{\sqrt{f(x)}}$$

- Define $A_a = \{y \colon y \in \Omega \text{ and } f(y) \leqslant a\}$. So, for all $y \in A_a$, we have $f(y) \leqslant a$. Therefore, for any $y \in A_a$, we get

$$d_T(x, y) \geqslant \frac{f(x) - a}{\sqrt{f(x)}}$$

Longest Increasing Subsequence IV

- Since, the inequality holds for all $y \in A_a$, we conclude that

$$d_T(x, A_a) \geqslant \frac{f(x) - a}{\sqrt{f(x)}}$$

- **Observation.** If $f(x) \geqslant a + E$, then

$$d_A(x, A_a) \geqslant \frac{E}{\sqrt{a + E}}$$

- So, we conclude that

$$\mathbb{P}\left[f(\mathbb{X}) \geqslant a + E\right] \leqslant \mathbb{P}\left[d_T(\mathbb{X}, A_a) \geqslant \frac{E}{\sqrt{a + E}}\right]$$

Longest Increasing Subsequence V

- Multiplying both sides by $\mathbb{P}[\mathbb{X} \in A_a]$, we get

$$\mathbb{P}[\mathbb{X} \in A_a] \cdot \mathbb{P}\left[f(\mathbb{X}) \geqslant a + E\right] \leqslant \mathbb{P}[\mathbb{X} \in A_a] \cdot \mathbb{P}\left[d_T(\mathbb{X}, A_a) \geqslant \frac{E}{\sqrt{a+E}}\right]$$
$$\leqslant \exp\left(-\frac{E^2}{4(a+E)}\right)$$

  The last inequality is due to Talagrand inequality.
- Let m be the median of the random variable $f(\mathbb{X})$
- Suppose we set $a = m$. Then, we have $\mathbb{P}[\mathbb{X} \in A_a] \geqslant 1/2$. Therefore, we conclude that

$$\mathbb{P}\left[f(\mathbb{X}) \geqslant m + E\right] \leqslant 2\exp\left(-\frac{E^2}{4(m+E)}\right)$$

  This concentration inequality implies a concentration radius of $E = \sqrt{n}$

Longest Increasing Subsequence VI

- Suppose we set $a + E = m$. Then, we have $\mathbb{P}\left[f(\mathbb{X}) \geqslant a_E\right] \geqslant 1/2$. s Then, we conclude

$$\mathbb{P}\left[\mathbb{X} \in A_a\right] = \mathbb{P}\left[f(\mathbb{X}) \leqslant m - E\right] \leqslant 2\exp\left(-\frac{E^2}{4m}\right)$$

Again, the radius of concentration is $\sqrt{m}$.

Configuration Function

- The approach of applying the Talagrand inequality to the problem of longest increasing subsequence can be generalized to several problems
- Consider the definition of c-configuration functions

Definition (Configuration Functions)

A function f is a c-configuration function, if for every x, y, there exists $\alpha_{x,y}$ such that the following holds

$$f(y) \geqslant f(x) - \sqrt{c \cdot f(x)}\, d_{\alpha_{x,y}}(x, y)$$

- Note that the longest increasing subsequence defines $f(\cdot)$ that is 1-configuration function. The derivation used above can be identically used for c-configuration functions