Lecture 16: Talagrand Inequality Application
Suppose $\mathbf{X} = (X_1, \ldots, X_n)$, where each $X_i$ is independent and uniformly distributed over $\Omega_i = [0, 1)$.

We are interested in demonstrating a concentration bound for $f(\mathbf{X})$, where $f(\mathbf{X})$ is the longest increasing subsequence in $(X_1, \ldots, X_n)$.

**Observation.** Consider any $x \in \Omega := \Omega_1 \times \cdots \times \Omega_n$. If $f(x) = k$ (i.e., the longest increased subsequence in $x$ is $k$), then there is a set $K_x = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ such that $K_x$ denotes the indices of the longest increasing subsequence in $x$.

**Observation.** Consider any $y \in \Omega$. Note that if $y$ agrees with $x$ at all the indices in $K_x$, then we have $f(y) \geq f(x)$ (it is possible that $y$ has a longest increasing subsequence, but, definitely, it will not be shorter than the length of the longest increasing subsequence in $x$).
**Observation.** Let us generalize the previous observation further. Consider any \( y \in \Omega \). Note that if \( y \) agrees with \( x \) at all indices in \( K_x \) except at \( \ell \) indices. Then, we have \( f(y) \geq f(x) - \ell \). Formally, we can write this as follows

\[
f(y) \geq f(x) - \left| \{ i : i \in K_x \text{ and } x_i \neq y_i \} \right|
\]

Intuitively, we incur a penalty for every \( i \in K_x \) where \( x \) and \( y \) differ. Let us fix \( \alpha_x = (\alpha_1, \ldots, \alpha_n) \) such that

\[
\alpha_i = \begin{cases} 
0 & i \notin K_x \\
\frac{1}{\sqrt{|K_x|}} & i \in K_x
\end{cases}
\]

Note that \( |K_x| = f(x) \). So, we conclude that

\[
f(y) \geq f(x) - \sqrt{f(x)} d_{\alpha_x}(x, y)
\]
Rearranging, we get that

\[ d_{\alpha_x}(x, y) \geq \frac{f(x) - f(y)}{\sqrt{f(x)}} \]

Since, \( d_T(\cdot, \cdot) \) is a supremum of \( d_\alpha(\cdot, \cdot) \) over all \( \alpha \) with norm-1, we get that

\[ d_T(x, y) \geq \frac{f(x) - f(y)}{\sqrt{f(x)}} \]

Define \( A_a = \{ y : y \in \Omega \text{ and } f(y) \leq a \} \). So, for all \( y \in A_a \), we have \( f(y) \leq a \). Therefore, for any \( y \in A_a \), we get

\[ d_T(x, y) \geq \frac{f(x) - a}{\sqrt{f(x)}} \]
Since, the inequality holds for all $y \in A_a$, we conclude that

$$d_T(x, A_a) \geq \frac{f(x) - a}{\sqrt{f(x)}}$$

Observation. If $f(x) \geq a + E$, then

$$d_A(x, A_a) \geq \frac{E}{\sqrt{a + E}}$$

So, we conclude that

$$\mathbb{P} [ f(X) \geq a + E ] \leq \mathbb{P} \left[ d_T(X, A_a) \geq \frac{E}{\sqrt{a + E}} \right]$$
Longest Increasing Subsequence V

- Multiplying both sides by $\mathbb{P} [X \in A_a]$, we get

$$\mathbb{P} [X \in A_a] \cdot \mathbb{P} [f(X) \geq a + E] \leq \mathbb{P} [X \in A_a] \cdot \mathbb{P} \left[ d_T(X, A_a) \geq \frac{E}{\sqrt{a + E}} \right]$$

$$\leq \exp \left( - \frac{E^2}{4(a + E)} \right)$$

The last inequality is due to Talagrand inequality.

- Let $m$ be the median of the random variable $f(X)$

- Suppose we set $a = m$. Then, we have $\mathbb{P} [X \in A_a] \geq 1/2$. Therefore, we conclude that

$$\mathbb{P} [f(X) \geq m + E] \leq 2 \exp \left( - \frac{E^2}{4(m + E)} \right)$$

This concentration inequality implies a concentration radius of $E = \sqrt{n}$
Suppose we set $a + E = m$. Then, we have
\[ \mathbb{P} \left[ f(X) \geq aE \right] \geq 1/2. \]
Then, we conclude
\[
\mathbb{P} \left[ X \in A_a \right] = \mathbb{P} \left[ f(X) \leq m - E \right] \leq 2 \exp \left( - \frac{E^2}{4m} \right)
\]
Again, the radius of concentration is $\sqrt{m}$. 

Talagrand Inequality
The approach of applying the Talagrand inequality to the problem of longest increasing subsequence can be generalized to several problems.

Consider the definition of $c$-configuration functions.

**Definition (Configuration Functions)**

A function $f$ is a $c$-configuration function, if for every $x, y$, there exists $\alpha_{x,y}$ such that the following holds

$$f(y) \geq f(x) - \sqrt{c \cdot f(x)} d_{\alpha_{x,y}}(x, y)$$

Note that the longest increasing subsequence defines $f(\cdot)$ that is $1$-configuration function. The derivation used above can be identically used for $c$-configuration functions.