Homework 0
(0 Points. Practice Problems only.)

1. **Approximating Exponentiation.** The goal of this problem is to approximate the function $\exp(-\varepsilon)$, where $0 \leqslant \varepsilon \leqslant 1$, using polynomials. We shall use the Lagrange form of the Taylor remainder theorem to perform this estimate.

   Let us define the function $f(x) = \exp(-x)$. Verify that $f^{(i)}(x) = (-1)^i \exp(-x)$, where $i \geqslant 0$ and $f^{(i)}$ is the short-had for the i-th derivative of the function f. Let us set $a = 0$.

   Recall that the Taylor's series of the function f is defined as follows.

   $$f(\varepsilon) = \sum_{i \geqslant 0} f^{(i)}(0) \frac{\varepsilon^i}{i!}$$

   For $f(x) = \exp(-x)$, we have

   $$\exp(-\varepsilon) = 1 - \varepsilon + \frac{\varepsilon^2}{2!} - \frac{\varepsilon^3}{3!} + \cdots$$

   Recall that the Lagrange form of the Taylor's remainder theorem is defined as follows. For every ε and $k \geqslant 0$, there exists $\theta \in (0,1)$ such that

   $$f(\varepsilon) = \left(\sum_{i \geqslant 0} f^{(i)}(0) \frac{\varepsilon^i}{i!} \right) + f^{(k+1)}(\theta\varepsilon) \frac{\varepsilon^{k+1}}{(k+1)!}$$

   Therefore, for our choice of $f(x) = \exp(-x)$, we have

   $$\exp(-\varepsilon) = \overbrace{\left(1 - \varepsilon + \frac{\varepsilon^2}{2!} - \frac{\varepsilon^3}{3!} + \cdots + (-1)^k \frac{\varepsilon^k}{k!} \right)}^{\text{Approximation: } p_k(\varepsilon)} + \overbrace{(-1)^{k+1} \exp(-\theta\varepsilon) \frac{\varepsilon^{k+1}}{(k+1)!}}^{\text{Remainder: } R_{\varepsilon,k}}$$

   (a) Prove that for even k, the remainder $R_{\varepsilon,k}$ is negative. Therefore, you have proved that $\exp(-\varepsilon) \leqslant p_k(\varepsilon)$.

   (b) Prove that for odd k, the remainder $R_{\varepsilon,k}$ is positive. Therefore, you have proved that $\exp(-\varepsilon) \geqslant p_k(\varepsilon)$.

   (c) Prove that the magnitude of the remainder $\left|R_{\varepsilon,k}\right| \leqslant \frac{\varepsilon^{k+1}}{(k+1)!}$. This proves a bound on the quality of the approximation of $\exp(-\varepsilon)$ by the polynomial $p_k(\varepsilon)$.

   (**Remark:** Students are encouraged to plot $\exp(-x)$ and the polynomials $p_k(x)$ to understand the bounds proved in this problem.)

2. **Approximating Logarithm.** The goal of this problem is to approximate the function $\ln(1-\varepsilon)$, where $0 \leqslant \varepsilon < 1$, using polynomials.

   Let us define $f(x) = \ln(1-x)$. Verify that $f^{(i)}(x) = -\frac{(i-1)!}{(1-x)^i}$, for $i \geqslant 1$.

   Verify that, for $f(x) = \ln(1-x)$, the Taylor series gives us

   $$\ln(1-\varepsilon) = -\varepsilon - \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{3} - \cdots$$

   Verify that, for $f(x) = \ln(1-x)$, the Lagrange form of the Taylor's remainder theorem gives us the following. For every ε, k, there exists $\theta \in (0,1)$ such that

   $$\ln(1-\varepsilon) = \overbrace{\left(-\varepsilon - \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{3} - \cdots - \frac{\varepsilon^k}{k}\right)}^{\text{Approximation: } p_k(\varepsilon)} - \frac{1}{(1-\theta\varepsilon)^{k+1}} \frac{\varepsilon^{k+1}}{(k+1)}$$

   (a) Prove that $\ln(1-\varepsilon) \leqslant p_k(\varepsilon)$, for all $k \geqslant 0$.

   (b) How large is the magnitude of the remainder as a function of k and ε?

   (c) Prove that $\ln(1-\varepsilon) \geqslant p_k(\varepsilon) - \frac{\varepsilon^k}{k}$, for all $0 \leqslant \varepsilon \leqslant 1/2$.
   (**Hint:** Use the fact that $\ln(1-\varepsilon) = -\varepsilon - \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{3} - \cdots$, for $0 \leqslant \varepsilon < 1$.)

   (**Remark:** Again, students are encouraged to plot $\ln(1-x)$ and the polynomials $p_k(x)$ and the polynomials $p_k(x) - \frac{x^k}{k}$ to understand the bounds proved in this problem.)

3. **AM-GM Inequality.** The goal of this problem is to prove the AM-GM inequality using the Jensen's inequality. Let us recall the Jensen's inequality. A function f is convex in the range $[a, b]$ if $f^{(2)}$ is positive in the range $[a, b]$. Jensen's inequality states that if f is convex in the range $[a, b]$, then

$$\frac{f(a)+f(b)}{2} \geqslant f\left(\frac{a+b}{2}\right)$$

Equality holds if and only if $a = b$.

A function f is concave in the range $[a, b]$ if $f^{(2)}$ is negative in the range $[a, b]$. Jensen's inequality states that if f is concave in the range $[a, b]$, then

$$\frac{f(a)+f(b)}{2} \leqslant f\left(\frac{a+b}{2}\right)$$

Equality holds if and only if $a = b$.

Let us recall the AM-GM inequality. For positive a, b, we have

$$\frac{a+b}{2} \geqslant \sqrt{ab}$$

Equality holds if and only if $a = b$.

Prove the AM-GM inequality using Jensen's inequality using $f(x) = \ln(x)$ (recall that $\ln(x)$ is concave in $(0, \infty)$).

4. **Cauchy-Schwarz Inequality.** The goal of this problem is to prove the Cauchy-Schwarz inequality using the Jensen's inequality. Let us recall the Cauchy-Schwarz inequality. For positive a_1, a_2, b_1, b_2, we have

$$(a_1 b_1 + a_2 b_2) \leqslant \left(a_1^2 + a_2^2\right)^{1/2} \left(b_1^2 + b_2^2\right)^{1/2}$$

Equality holds if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$.

(a) Let us consider an intermediate inequality. For positive A, B, we have

$$\left(1 + \sqrt{AB}\right) \leqslant (1 + A)^{1/2} (1 + B)^{1/2}$$

Equality holds if and only if $A = B$.

Use this intermediate inequality to prove the Cauchy-Schwarz inequality.

(b) Prove that the function $f(x) = \ln\left(1 + \exp(x)\right)$ is a convex function.

(c) Prove the intermediate inequality using the Jensen's inequality on the function $f(x) = \ln\left(1 + \exp(x)\right)$.

5. **Young's Inequality.** The goal of this problem is to prove the Young's inequality using the (general form) of the Jensen's inequality. One can interpret Young's inequality as a generalization of the AM-GM inequality.

   Let us recall the general form of the Jensen's inequality. Suppose f is a convex function in the range $[a, b]$. For any positive α, β such that $\alpha + \beta = 1$, we have

   $$\alpha f(a) + \beta f(b) \geqslant f(\alpha a + \beta b)$$

   Equality holds if and only if $a = b$.

   Note that if we choose $\alpha = \beta = \frac{1}{2}$, we get the particular form of Jensen's inequality as used in the previous two problems.

   Let us now recall the Young's inequality. Let p, q be Hölder conjugates, i.e., positive reals numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. For positive a, b, we have

   $$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$$

   Equality holds if and only if $a^p = b^q$.

   Prove Young's inequality using the general form of the Jensen's inequality on the function $f(x) = \ln(x)$.

   (**Remark.** Note that for $p = q = 2$, Young's inequality is identical to the AM-GM inequality.)

6. **Hölder's Inequality.** The goal of the problem is to prove the Hölder's inequality using the (general form) of the Jensen's inequality.

   Let us recall Hölder's inequality. Suppose p, q are Hölder conjugates. For positive a_1, a_2, b_1, b_2, we have

   $$(a_1 b_1 + a_2 b_2) \leqslant \left(a_1^p + a_2^p\right)^{1/p} \left(b_1^q + b_2^q\right)^{1/q}$$

   (a) Consider the following intermediate inequality. For positive A, B, we have

   $$\left(1 + A^{1/p} B^{1/q}\right) \leqslant (1 + A)^{1/p} (1 + B)^{1/q}$$

   Prove the Hölder's inequality using the intermediate inequality.

   (b) Prove the intermediate inequality using the general form of the Jensen's inequality on the function $f(x) = \ln(1 + \exp(x))$.

   (c) What is the characterization of achieving equality in the Hölder's inequality?