Homework 0
(0 Points. Practice Problems only.)

1. **Approximating Exponentiation.** The goal of this problem is to approximate the function $\exp(-\varepsilon)$, where $0 \leq \varepsilon \leq 1$, using polynomials. We shall use the Lagrange form of the Taylor remainder theorem to perform this estimate.

Let us define the function $f(x) = \exp(-x)$. Verify that $f^{(i)}(x) = (-1)^i \exp(-x)$, where $i \geq 0$ and $f^{(i)}$ is the short-hand for the $i$-th derivative of the function $f$. Let us set $a = 0$.

Recall that the Taylor’s series of the function $f$ is defined as follows.

$$f(\varepsilon) = \sum_{i \geq 0} f^{(i)}(0) \frac{\varepsilon^i}{i!}$$

For $f(x) = \exp(-x)$, we have

$$\exp(-\varepsilon) = 1 - \varepsilon + \frac{\varepsilon^2}{2!} - \frac{\varepsilon^3}{3!} + \cdots$$

Recall that the Lagrange form of the Taylor’s remainder theorem is defined as follows. For every $\varepsilon$ and $k \geq 0$, there exists $\theta \in (0, 1)$ such that

$$f(\varepsilon) = \left(\sum_{i \geq 0} f^{(i)}(0) \frac{\varepsilon^i}{i!}\right) + f^{(k+1)}(\theta \varepsilon) \frac{\varepsilon^{k+1}}{(k+1)!}$$

Therefore, for our choice of $f(x) = \exp(-x)$, we have

$$\exp(-\varepsilon) = \left(1 - \varepsilon + \frac{\varepsilon^2}{2!} - \frac{\varepsilon^3}{3!} + \cdots + (-1)^k \frac{\varepsilon^k}{k!}\right) + (-1)^{k+1} \exp(-\theta \varepsilon) \frac{\varepsilon^{k+1}}{(k+1)!}$$

(a) Prove that for even $k$, the remainder $R_{\varepsilon,k}$ is negative. Therefore, you have proved that $\exp(-\varepsilon) \leq p_k(\varepsilon)$.

(b) Prove that for odd $k$, the remainder $R_{\varepsilon,k}$ is positive. Therefore, you have proved that $\exp(-\varepsilon) \geq p_k(\varepsilon)$.

(c) Prove that the magnitude of the remainder $|R_{\varepsilon,k}| \leq \frac{\varepsilon^{k+1}}{(k+1)!}$. This proves a bound on the quality of the approximation of $\exp(-\varepsilon)$ by the polynomial $p_k(\varepsilon)$.

**Remark:** Students are encouraged to plot $\exp(-x)$ and the polynomials $p_k(x)$ to understand the bounds proved in this problem.
2. Approximating Logarithm. The goal of this problem is to approximate the function \[ \ln(1 - \varepsilon), \] where \( 0 \leq \varepsilon < 1 \), using polynomials.

Let us define \( f(x) = \ln(1 - x) \). Verify that \( f^{(i)}(x) = -\frac{(i-1)!}{(1-x)^i}, \) for \( i \geq 1 \).

Verify that, for \( f(x) = \ln(1 - x) \), the Taylor series gives us

\[ \ln(1 - \varepsilon) = -\varepsilon - \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{3} - \cdots \]

Verify that, for \( f(x) = \ln(1 - x) \), the Lagrange form of the Taylor’s remainder theorem gives us the following. For every \( \varepsilon, k \), there exists \( \theta \in (0, 1) \) such that

\[ \ln(1 - \varepsilon) = \left( -\varepsilon - \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{3} - \cdots - \frac{\varepsilon^k}{k} \right) - \frac{1}{(1 - \theta \varepsilon)^{k+1}} \frac{\varepsilon^{k+1}}{(k + 1)} \]

(a) Prove that \( \ln(1 - \varepsilon) \leq p_k(\varepsilon) \), for all \( k \geq 0 \).

(b) How large is the magnitude of the remainder as a function of \( k \) and \( \varepsilon \)?

(c) Prove that \( \ln(1 - \varepsilon) \geq p_k(\varepsilon) - \frac{\varepsilon^k}{k} \), for all \( 0 \leq \varepsilon \leq 1/2 \).

(Hint: Use the fact that \( \ln(1 - \varepsilon) = -\varepsilon - \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{3} - \cdots \), for \( 0 \leq \varepsilon < 1 \).

(Remark: Again, students are encouraged to plot \( \ln(1 - x) \) and the polynomials \( p_k(x) \) and the polynomials \( p_k(x) - \frac{\varepsilon^k}{k} \) to understand the bounds proved in this problem.)
3. **AM-GM Inequality.** The goal of this problem is to prove the AM-GM inequality using the Jensen’s inequality. Let us recall the Jensen’s inequality. A function $f$ is convex in the range $[a, b]$ if $f''(x)$ is positive in the range $[a, b]$. Jensen’s inequality states that if $f$ is convex in the range $[a, b]$, then

$$
\frac{f(a) + f(b)}{2} \geq f \left( \frac{a + b}{2} \right)
$$

Equality holds if and only if $a = b$.

A function $f$ is concave in the range $[a, b]$ if $f''(x)$ is negative in the range $[a, b]$. Jensen’s inequality states that if $f$ is concave in the range $[a, b]$, then

$$
\frac{f(a) + f(b)}{2} \leq f \left( \frac{a + b}{2} \right)
$$

Equality holds if and only if $a = b$.

Let us recall the AM-GM inequality. For positive $a, b$, we have

$$
\frac{a + b}{2} \geq \sqrt{ab}
$$

Equality holds if and only if $a = b$.

Prove the AM-GM inequality using Jensen’s inequality using $f(x) = \ln(x)$ (recall that $\ln(x)$ is concave in $(0, \infty)$).
4. **Cauchy-Schwarz Inequality.** The goal of this problem is to prove the Cauchy-Schwarz inequality using the Jensen’s inequality. Let us recall the Cauchy-Schwarz inequality. For positive $a_1, a_2, b_1, b_2$, we have

$$ (a_1b_1 + a_2b_2) \leq \left( a_1^2 + a_2^2 \right)^{1/2} \left( b_1^2 + b_2^2 \right)^{1/2} $$

Equality holds if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$.

(a) Let us consider an intermediate inequality. For positive $A, B$, we have

$$ \left( 1 + \sqrt{AB} \right) \leq (1 + A)^{1/2} (1 + B)^{1/2} $$

Equality holds if and only if $A = B$.

Use this intermediate inequality to prove the Cauchy-Schwarz inequality.

(b) Prove that the function $f(x) = \ln(1 + \exp(x))$ is a convex function.

(c) Prove the intermediate inequality using the Jensen's inequality on the function $f(x) = \ln(1 + \exp(x))$. 


5. **Young’s Inequality.** The goal of this problem is to prove the Young’s inequality using the (general form) of the Jensen’s inequality. One can interpret Young’s inequality as a generalization of the AM-GM inequality.

Let us recall the general form of the Jensen’s inequality. Suppose $f$ is a convex function in the range $[a, b]$. For any positive $\alpha, \beta$ such that $\alpha + \beta = 1$, we have

$$\alpha f(a) + \beta f(b) \geq f(\alpha a + \beta b)$$

Equality holds if and only if $a = b$.

Note that if we choose $\alpha = \beta = \frac{1}{2}$, we get the particular form of Jensen’s inequality as used in the previous two problems.

Let us now recall the Young’s inequality. Let $p, q$ be Hölder conjugates, i.e., positive reals numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. For positive $a, b$, we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Equality holds if and only if $a^p = b^q$.

Prove Young’s inequality using the general form of the Jensen’s inequality on the function $f(x) = \ln(x)$.

(\textbf{Remark.} Note that for $p = q = 2$, Young’s inequality is identical to the AM-GM inequality.)
6. Hölder’s Inequality. The goal of the problem is to prove the Hölder’s inequality using the (general form) of the Jensen’s inequality.

Let us recall Hölder’s inequality. Suppose $p, q$ are Hölder conjugates. For positive $a_1, a_2, b_1, b_2$, we have

$$(a_1b_1 + a_2b_2) \leq (a_1^p + a_2^p)^{1/p} (b_1^q + b_2^q)^{1/q}$$

(a) Consider the following intermediate inequality. For positive $A, B$, we have

$$\left(1 + A^{1/p}B^{1/q}\right) \leq (1 + A)^{1/p} (1 + B)^{1/q}$$

Prove the Hölder’s inequality using the intermediate inequality.

(b) Prove the intermediate inequality using the general form of the Jensen’s inequality on the function $f(x) = \ln(1 + \exp(x))$.

(c) What is the characterization of achieving equality in the Hölder’s inequality?