Lecture 25: Hypercontractivity and Parity over Large Sets
For $p \geq 1$, let us define the $p$-norm of a function

$$
\|f\|_p = \left( \frac{1}{N} \sum_{x \in \{0,1\}^n} |f(x)|^p \right)^{1/p}
$$

We can use Jensen’s inequality to prove that $\|f\|_p \leq \|f\|_q$ for any function $f$, when $p < q$, and equality holds if and only if $|f|$ is a constant function.
Basic Observations

We can show, using Jensen’s inequality, that \( \| T_\rho(f) \|_p \leq \| f \|_p \) (Intuition: because the noise operator smoothens the function)

That is, “\( T_\rho(\cdot) \) contracts \( f \)”

By monotonicity of norm, we can conclude that
\[ \| T_\rho(f) \|_p \leq \| T_\rho(f) \|_q, \text{ for } p \leq q \]

So, we summarize the above discussion using the following picture

\[
\begin{align*}
\| T_\rho(f) \|_p & \leq \| f \|_p \\
\wedge \\
\| T_\rho(f) \|_q
\end{align*}
\]

But how does \( \| f \|_p \) relate to \( \| T_\rho(f) \|_q \)?
Theorem (Hypercontractivity Theorem)

For $1 \leq p \leq q$ we have

$$\| T_\rho(f) \|_q \leq \| f \|_p,$$

for all $0 \leq \rho \leq \sqrt{(p - 1)/(q - 1)}$.

- Intuitively, the hypercontractivity theorem states that even the $q$-th norm of $T_\rho(f)$ is smaller than the $p$-th norm of $f$, if $q$ is slightly larger than $p$.
- The tightest inequality is obtained for $\rho = \sqrt{(p - 1)/(q - 1)}$. 
Special Case: $q = 2$

- Suppose $p = 1 + \delta$ and $q = 2$, then the hypercontractivity theorem states that
  \[ \| T_\rho(f) \|_2 \leq \| f \|_{1+\delta}, \]
  for $\rho = \sqrt{\delta}$
- By Parseval’s identity, we have
  \[ \| T_\rho(f) \|_2^2 = \sum_{S \in \{0,1\}^n} \delta^{|S|} \hat{f}(S)^2 \]
- So, we conclude the following result
  \[ \sum_{S \in \{0,1\}^n} \delta^{|S|} \hat{f}(S)^2 \leq \| f \|_{1+\delta}^2 \]
- Comment: Proving the hypercontractivity theorem for $1 \leq p \leq q = 2$ suffices to prove the general hypercontractivity theorem presented above
Application: KKL Lemma

- KKL stands for Kahn-Kalai-Linial
- Suppose $f : \{0, 1\}^n \to \{-1, 0, +1\}$
- Then, we have $\|f\|_p = \mathbb{P}[f \neq 0]^{1/p}$
- By direct application of the hypercontractivity theorem, we can conclude that

**Lemma (KKL Lemma)**

$$\sum_{S \subseteq \{0, 1\}^n} \delta^{|S|} \widehat{f}(S)^2 \leq \mathbb{P}[f \neq 0]^{2/(1+\delta)}$$

- Intuition: The left-hand side is dominated by the Fourier coefficients associated with “small-weight $S$.” So, the inequality states that the “total mass associated with the Fourier coefficients of small-weight $S$” is much smaller than the probability of “encountering $f$.” Equivalently, a “small support $f$” has low “total mass” on the Fourier coefficients of small $S$. 

Fourier Analysis
Suppose $A \subseteq \{0, 1\}^n$ and $\mathbf{1}_A$ is the indicator function of the set $A$

For $S \in \{0, 1\}^n$, define

$$\beta_S = \frac{1}{|A|} \sum_{x \in A} \chi_S(x)$$

Intuitively, the quantity $\beta_S$ represents “how random the set $A$ appears” when we perform the test $\chi_S$. Smaller $\beta_S$ implies more random $A$ appears.

We can perform the following manipulation

$$\beta_S = \frac{1}{|S|} \sum_{x \in A} \chi_S(x) = \frac{1}{|A|} \sum_{x \in \{0, 1\}^n} \mathbf{1}_A(x) \chi_S(x) = \frac{N}{|A|} \langle \mathbf{1}_A, \chi_S \rangle = \frac{N}{|A|} \mathbf{1}_A(S)$$
Our goal is to study the following quantity

\[ \sum_{S \in \{0,1\}^n : |S| = k} \beta_S^2 \]

Note that this quantity is the cumulative measure of randomness of all \( \chi_S \) test such that \( |S| = k \).

Now, we can use the expression of \( \beta_S \) to obtain

\[ \sum_{S \in \{0,1\}^n : |S| = k} \beta_S^2 = \frac{N^2}{|A|^2} \sum_{S \in \{0,1\}^n : |S| = k} 1_A(S)^2 \]

The KKL Lemma provides us the perfect tool to upper-bound the right hand side. For any \( \delta \in [0,1] \), we have

\[ \sum_{S \in \{0,1\}^n : |S| = k} \delta^k \hat{f}(S)^2 \leq \sum_{S \in \{0,1\}^n} \delta^{|S|} \hat{f}(S)^2 \leq P[f \neq 0]^2/(1+\delta) \]
So, we obtain

\[
\beta^2 = \frac{N^2}{|A|^2} \sum_{S \in \{0,1\}^n : |S|=k} 1_{\{A\}}(S) \leq \frac{N^2}{|A|^2} \cdot \frac{1}{\delta^k} \cdot \mathbb{P} \left[ 1_{\{A\}} \neq 0 \right]^{2/(1+\delta)}\]

\[
= \frac{N^2}{|A|^2} \cdot \frac{1}{\delta^k} \cdot \left( \frac{|A|}{N} \right)^{2/(1+\delta)}
\]

\[
= \left( \frac{N}{|A|} \right)^{2 \left(1-\frac{1}{1+\delta} \right)} \cdot \frac{1}{\delta^k} = \left( \frac{N}{|A|} \right)^{\frac{2\delta}{1+\delta}} \cdot \frac{1}{\delta^k}
\]

\[
\leq \left( \frac{N}{|A|} \right)^{2\delta} \cdot \frac{1}{\delta^k}
\]
Now, our objective is to find $\delta \in [0, 1]$ that minimizes the right hand side expression. This part is left as an exercise.

At the end, for this value of $\delta$, we shall have

$$\left( \binom{n}{k} \right)^{-1} \sum_{S \in \{0,1\}^n : |S|=k} \beta_S^2 = O \left( 1 - \frac{a}{n} \right)^k,$$

where $|A| = 2^a$. That is, the average bias is exponentially small. This bound is also (essentially) tight.