Lecture 19: Discrete Fourier Analysis on the Boolean Hypercube (Introduction)
Recall

- Our objective is to study function $f : \{0, 1\}^n \rightarrow \mathbb{R}$
- Every function $f$ is equivalently represented as the vector $(f(0), f(1), \ldots, f(N - 1)) \in \mathbb{R}^N$, where $N = 2^n$
- For $S = S_1 S_2 \ldots S_n \in \{0, 1\}^n$, define the function

$$\chi_S(x) = (-1)^{S_1 x_1 + S_2 x_2 + \cdots + S_n x_n},$$

where $x = x_1 x_2 \ldots x_n$
- We defined an inner-product of functions

$$\langle f, g \rangle := \frac{1}{N} \sum_{x \in \{0, 1\}^n} f(x)g(x)$$

- We showed that $\chi_S$ are orthonormal, i.e.,

$$\langle \chi_S, \chi_T \rangle = \begin{cases} 0, & \text{if } S \neq T \\ 1, & \text{if } S = T \end{cases}$$
Since \( \{ \chi_S : S \in \{0, 1\}^n \} \) is an orthonormal basis, we can express any \( f \) as follows

\[
f = \hat{f}(0)\chi_0 + \hat{f}(1)\chi_1 + \cdots + \hat{f}(N-1)\chi_{N-1},
\]

where \( \hat{f}(S) \in \mathbb{R} \) and \( S \in \{0, 1\}^n \)

We interpret \((\hat{f}(0), \hat{f}(1), \ldots, \hat{f}(N-1))\) as a function \( \hat{f} \)
Fourier Transformation

- Fourier Transformation is a basis change that maps $f$ to $\hat{f}$.
- We shall represent it as $f \overset{\mathcal{F}}{\mapsto} \hat{f}$, where $\mathcal{F}$ is the Fourier Transformation.
Linearity of Fourier Transformation I

- Note that we have the following property. For any $S \in \{0, 1\}^n$, we have

$$(f(0) \ f(1) \cdots \ f(N - 1)) \cdot \frac{1}{N} \ (\chi_S(0) \ \chi_S(1) \cdots \ \chi_S(N - 1))^T = \hat{f}(S)$$

- Define the matrix

$$F = \frac{1}{N} \begin{bmatrix}
\chi_0(0) & \chi_1(0) & \cdots & \chi_{N-1}(0) \\
\chi_0(1) & \chi_1(1) & \cdots & \chi_{N-1}(1) \\
\vdots & \vdots & \ddots & \vdots \\
\chi_0(N - 1) & \chi_1(N - 1) & \cdots & \chi_{N-1}(N - 1)
\end{bmatrix}$$

- From the property mentioned above, we have $f \cdot F = \hat{f}$
Claim

For two function $f, g : \{0, 1\}^n \rightarrow \mathbb{R}$, we have

$$(f + g) = \hat{f} + \hat{g}$$

Proof.

$$(f + g) = (f + g)\mathcal{F} = f\mathcal{F} + g\mathcal{F} = \hat{f} + \hat{g}$$
Claim

For a function $f : \{0, 1\}^n \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$, we have

$$\hat{(cf)} = c\hat{f}$$

Proof.

$$\hat{cf} = (cf)\mathcal{F} = c(f\mathcal{F}) = c\hat{f} \qed$$
Theorem

Let $f : \{0, 1\}^n \to \mathbb{R}$. Then, we have

$$\hat{f} = \frac{1}{N} \cdot f$$

Proof.

- We shall prove that $\mathcal{F} \cdot \mathcal{F} = \frac{1}{N} I_{N \times N}$. This result shall imply that $$\hat{f} = (f \mathcal{F}) \mathcal{F} = f \left( \frac{1}{N} I_{N \times N} \right) = \frac{1}{N} I_{N \times N}$$

- Let us compute the element $(\mathcal{F} \mathcal{F})_{i,j}$. This element is the product of the $i$-th row of $\mathcal{F}$ and the $j$-th column of $\mathcal{F}$

- The $j$-th column of $\mathcal{F}$ is $$\left( \frac{1}{N} \chi_j \right)^T$$

- The $i$-th row of $\mathcal{F}$ is $$(\chi_0(i) \chi_1(i) \cdots \chi_{N-1}(i))$$

- Note that $\chi_S(x) = \chi_x(S)$, i.e., the matrix $\mathcal{F}$ is symmetric

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Fourier Analysis
So, the $i$-th row of $\mathcal{F}$ is $\frac{1}{N}\chi_i$

Therefore, we have $(\mathcal{F}\mathcal{F})_{i,j} = \frac{1}{N^2} \cdot \chi_i \cdot \chi_j^\top = \frac{1}{N} \langle \chi_i, \chi_j \rangle$. The orthonormality of the Fourier basis completes the proof.
Suppose \( f, g : \{0,1\}^n \rightarrow \mathbb{R} \). Then, the following holds

\[
\langle f, g \rangle = \sum_{S \in \{0,1\}^n} \hat{f}(S)\hat{g}(S)
\]
Proof.

\[
\langle f, g \rangle = \left\langle \sum_{S \in \{0, 1\}^n} \hat{f}(S) \chi_S, \sum_{T \in \{0, 1\}^n} \hat{g}(T) \chi_T \right\rangle
\]

\[
= \sum_{S \in \{0, 1\}^n} \hat{f}(S) \left\langle \chi_S, \sum_{T \in \{0, 1\}^n} \hat{g}(T) \chi_T \right\rangle
\]

\[
= \sum_{S \in \{0, 1\}^n} \hat{f}(S) \sum_{T \in \{0, 1\}^n} \langle \chi_S, \chi_T \rangle
\]

\[
= \sum_{S \in \{0, 1\}^n} \hat{f}(S) \hat{g}(S)
\]

Note that, if \( f, g : \{0, 1\}^n \rightarrow \{+1, -1\} \) then we have \( \langle f, g \rangle = 1 - \varepsilon \), there \( f \) and \( g \) disagree at \( \varepsilon N \) inputs. Intuitively, if \( |\langle f, g \rangle| \) is close to 1 then the functions are highly correlated. On the other hand, if \( |\langle f, g \rangle| \) is close to 0 then the functions are independent.
Suppose $f : \{0, 1\}^n \rightarrow \mathbb{R}$. Then

$$\langle f, g \rangle = \sum_{S \in \{0, 1\}^n} \hat{f}(S)^2$$

Substitute $f = g$ in Plancherel’s theorem.
Corollary

If $f : \{0, 1\}^n \to \{+1, -1\}$, then $\sum_{S \in \{0, 1\}^n} \hat{f}(S)^2 = 1$

Follows from the fact that $\langle f, f \rangle = 1$ and Parseval’s identity.