Homework 1

1. (4+1 + 4+1 points) Use Jensen’s Inequality to prove the following inequalities.

- Let $A, B$ and $C$ be the three angles of a triangle. Prove that $\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$.
- Find the exact criterion for equality.

- Let $X$ be a random variable over a finite sample space $\Omega$. We define the entropy of the random variable $X$ as $H(X) := -\sum_{x \in \Omega} P[X = x] \lg P[X = x]$. Prove that $H(X) \leq \lg |\Omega|$. Find the exact criterion for equality.

2. (10 points) Let $M$ be a $n \times m$ matrix such that each of its entries is 0 or 1. Let $B = \{(i,j) : i \in [n], j \in [m], M(i,j) = 1\}$. Suppose $|B| \geq \varepsilon \cdot (nm)$, i.e., the set $B$ covers at least an $\varepsilon$ fraction of the entries of the matrix. Let $R$ be the set of rows $i$ such that $\sum_{j=1}^{m} M(i,j) \geq (\varepsilon/2) \cdot m$. Intuitively, $R$ is the set of rows in the matrix where at least $\varepsilon/2$ fraction of the entries at in the set $B$. Prove that $|R| \geq (\varepsilon/2)n$.

3. (10 points) Let $\pi(x)$ represent the number of prime numbers that are less than $x$. For example, $\pi(16) = \lvert \{2, 3, 5, 7, 11, 13\} \rvert = 6$. The celebrated “Prime Number Theorem” shows that $\pi(x)$ roughly behaves like $x/\lg x$, for large enough $x$. In this question assume that $\pi(x) = x/\lg x$.

Let $r(x)$ represent the number of bits that are needed to represent the number $x$ in binary. For example, $r(9) = 4$.

Let $\Omega_n = \{p: p$ is a prime, and $r(p) \leq n\}$. Let $X_n$ be uniform distribution over $\Omega_n$. What is $E[r(X_n)]$?

4. (10 points) Let $f(x)$ be a convex upward function for $x \geq 1$. Prove that

$$\sum_{i=1}^{n} f(i) \leq \frac{f(1) + f(n)}{2} + \int_{1}^{n} f(x) \, dx$$

5. (3 + 3 + 2 + 5 + 2 points) The “Stirling Approximation” states that $x!$ is roughly $\sqrt{2\pi x} \left(\frac{x}{e}\right)^x$, for large enough $x$. So, this approximation cannot be applied to small values of $x$.

Let $p \in (0, 1/2)$ be a constant. Let $B(n, p)$ be the number of $n$-bit binary strings that have at most $pn$ 1s. Note that we have:

$$B(n, p) = \sum_{0 \leq i \leq pn} \binom{n}{i}$$

We will upper bound this number.

(a) Prove that, for $1 \leq i \leq pn$, we have:

$$\binom{n}{i-1} \leq \frac{p}{1-p} \binom{n}{i}$$
(b) Prove that, for \( 0 \leq k \leq pn \), we have:

\[
\binom{n}{pn - k} \leq \left(\frac{p}{1 - p}\right)^k \binom{n}{pn}
\]

(c) Prove that

\[
B(n, p) \leq \left(\frac{1 - p}{1 - 2p}\right) \binom{n}{pn}
\]

(d) Let \( h: [0, 1] \to [0, 1] \) be the binary entropy function defined as follows:

\[
h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)
\]

Use Stirling Approximation to show that

\[
\binom{n}{pn} \text{ is roughly } \frac{2^{h(p)n}}{\sqrt{2p(1 - p)\pi n}}
\]

(e) Prove that, for large enough \( n \), \( B(n, p) \leq 2^{h(p)n} \).

6. (Extra Credit) Suppose we are given a black-box \( B \). If we ping this black-box, it outputs a uniformly random real number in the range \([0, 1]\).

Let \( S_n \) denote the set of all permutations of size \( n \) (A size \( n \) permutation is a bijection from \([n] \to [n]\)). We are interested in constructing an algorithm that uses this black-box to output a permutation that is drawn uniformly at random from \( S_n \). Consider the following conjectured approach:

```plaintext
function Uniform_Permutation_Generator(n)
for i = 1 to n do
    Ping B to obtain to obtain a sample \( x_i \)
end for
Let \( (x_{i_1}, \ldots, x_{i_n}) \) be the sorting of the sequence \( (x_1, \ldots, x_n) \) in an increasing order. (If there are collisions, i.e. there are \( i < j \) such that \( x_i = x_j \), then the sorted sequence has \( x_i \) before \( x_j \))
Let \( \pi \) be the permutation such that, for \( k \in [n] \), we have \( \pi(k) = i_k \).
Output \( \pi \)
end function
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Let \( X_n \) be the random variable corresponding to the output of the algorithm described above. Prove or disprove that, for any \( \pi \in S_n \), we have

\[
\mathbb{P}[X_n = \pi] = \frac{1}{n!}
\]