Lecture 18: Discrete Fourier Analysis on the Boolean Hypercube (Introduction)

Focus of Study

- Functions with domain $\{0,1\}^n$ and range $\mathbb{R}$
- Let $f\colon \{0,1\}^n \to \mathbb{R}$
- We will always use $N = 2^n$
- The n-bit binary strings will be canonically interpreted as the integer in the range $\{0,\dots,N-1\}$
- We equivalently interpret f as a vector in $\mathbb{R}^n$

$$\big(f(0), f(1), \dots, f(N-1)\big)$$

Basis Functions

- For $S \in \{0,1\}^n$, we will define the following function:

$$\chi_S(x) := (-1)^{\sum_{i=1}^{n} S_i x_i}$$

- Note that S is generally interpreted as a subset of $\{1, \ldots, n\}$. But the interpretation presented here is equivalent. I personally prefer this because it generalizes to other function domains.

Inner Product

Definition

The inner-product of two functions f and g is defined as follows:

$$\langle f, g \rangle := \frac{1}{N} \sum_{x \in \{0,1\}^n} f(x) g(x)$$

A Property of Basis Functions

Claim

$$\sum_{x \in \{0,1\}^n} \chi_R(x) = \begin{cases} 0, & \textit{if } R \neq \emptyset \\ N, & \textit{if } R = \emptyset \end{cases}$$

Proof.

For $R = \emptyset$, the proof is trivial. Assume that $R = \{i_1, \ldots, i_r\}$, where $r \geqslant 1$.

$$\begin{aligned}
\sum_{x \in \{0,1\}^n} \chi_R(x) &= \sum_{x_1 \in \{0,1\}} \cdots \sum_{x_1 \in \{0,1\}} (-1)^{\sum_{k=1}^{r} x_{i_k}} \\
&= \frac{N}{2^r} \sum_{x_{i_1} \in \{0,1\}} \cdots \sum_{x_{i_r} \in \{0,1\}} (-1)^{\sum_{k=1}^{r} x_{i_k}} \\
&= \frac{N}{2^r} \sum_{(x_{i_1},\ldots,x_{i_{r-1}}) \in \{0,1\}^{r-1}} (-1)^{\sum_{k=1}^{r-1} x_{i_k}} \sum_{x_{i_r} \in \{0,1\}} (-1)^{x_{i_r}} \\
&= \frac{N}{2^r} \sum_{(x_{i_1},\ldots,x_{i_{r-1}}) \in \{0,1\}^{r-1}} (-1)^{\sum_{k=1}^{r-1} x_{i_k}} \cdot 0 \\
&= 0
\end{aligned}$$

□

Orthonormality of Basis Functions

Lemma

$$\langle \chi_S, \chi_T \rangle = \begin{cases} 0, & \textit{if } S \neq T \\ 1, & \textit{if } S = T \end{cases}$$

We use the previous claim to prove this result.

Proof.

$$\begin{aligned} \langle \chi_S, \chi_T \rangle &= \frac{1}{N} \sum_{x \in \{0,1\}^n} \chi_S(x) \chi_T(x) \\ &= \frac{1}{N} \sum_{x \in \{0,1\}^n} \chi_R(x), \text{ where } R = S \Delta T \\ &= \begin{cases} 0, & \text{if } S \neq T \\ 1, & \text{if } S = T \end{cases} \end{aligned}$$

□

Fourier Coefficients

- Given $f\colon \{0,1\}^n \to \mathbb{R}$
- We define a new function $\widehat{f}\colon \{0,1\}^n \to \mathbb{R}$

$$\widehat{f}(S) := \langle f, \chi_S \rangle$$

- The Fourier Transform maps f to $\widehat{f}$

Fourier Transform is a Linear Bijection

- Let

$$\mathcal{F} = \frac{1}{N}\begin{pmatrix} \chi_0(0) & \chi_1(0) & \dots & \chi_{N-1}(0) \\ \chi_0(1) & \chi_1(1) & \dots & \chi_{N-1}(1) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_0(N-1) & \chi_1(N-1) & \dots & \chi_{N-1}(N-1) \end{pmatrix}$$

- Verify that $f \cdot \mathcal{F} = \widehat{f}$
- This proves that the Fourier Transform is a linear mapping
- To prove that this is a bijection, prove that $\mathcal{F} \cdot (N\mathcal{F}) = I_{N\times N}$

Fourier Representation

- We can write f as a linear combination of the Fourier Basis Functions
- So, we have

$$f = \sum_{S \in \{0,1\}^n} \widehat{f}(S)\chi_S$$

Inner-product of Functions

Lemma

$$\langle f, g \rangle = \sum_{S \in \{0,1\}^n} \widehat{f}(S)\widehat{g}(S)$$

Proof.

$$\begin{aligned}
\langle f, g \rangle &= \left\langle \sum_{S \in \{0,1\}^n} \widehat{f}(S)\chi_S, \sum_{T \in \{0,1\}^n} \widehat{g}(T)\chi_T \right\rangle \\
&= \sum_{S \in \{0,1\}^n} \widehat{f}(S) \left\langle \chi_S, \sum_{T \in \{0,1\}^n} \widehat{g}(T)\chi_T \right\rangle \\
&= \sum_{S,T \in \{0,1\}^n} \widehat{f}(S)\widehat{g}(T)\langle \chi_S, \chi_T \rangle \\
&= \sum_{S \in \{0,1\}^n} \widehat{f}(S)\widehat{g}(S)
\end{aligned}$$

Parseval's Identity

Lemma

$$\frac{1}{N}\sum_{x\in\{0,1\}^n} f(x)^2 = \sum_{S\in\{0,1\}^n} \widehat{f}(S)^2$$

Follows from the previous expression for $\langle f, f \rangle$

Consequences of Linearity of the Fourier Transform

- $\widehat{(cf)} = c\widehat{f}$
- $\widehat{(f+g)} = \widehat{f} + \widehat{g}$

Fourier of a Fourier

Lemma

$$\widehat{(\widehat{f})} = \frac{1}{N} f$$

Follows from the fact that $\mathcal{F} \cdot \mathcal{F} = \frac{1}{N} I_{N \times N}$

Fourier of an offset

Lemma

Suppose $f(x) = g(x + c)$*, for all* $x \in \{0, 1\}^n$*. Then* $\widehat{f}(S) = \chi_c(S)\widehat{g}(S)$*, for all* $S \in \{0, 1\}^n$*.*

Proof.

$$
\begin{aligned}
\langle f, \chi_S \rangle &= \frac{1}{N} \sum_{x \in \{0,1\}^n} f(x)\chi_S(x) \\
&= \frac{1}{N} \sum_{x \in \{0,1\}^n} g(x + c)\chi_S(x) \\
&= \frac{1}{N}\chi_S(c) \sum_{x \in \{0,1\}^n} g(x + c)\chi_S(x + c) \\
&= \chi_c(S)\widehat{g}(S)
\end{aligned}
$$

□