Lecture 11: Generalized Lovász Local Lemma

Recall

- We design an experiment with independent random variables $\mathbb{X}_1, \ldots, \mathbb{X}_m$
- We define bad events $\mathbb{A}_1, \ldots, \mathbb{A}_n$ where the bad event $\mathbb{A}_i$ depends on the variables $\left(\mathbb{X}_{k_1}, \ldots, \mathbb{X}_{k_{n_i}}\right)$
- We define $\mathsf{Vbl}_i = \{k_1, \ldots, k_{n_i}\}$, the set of all variables that the bad event $\mathbb{A}_i$ depends on
- The bad event $\mathbb{A}_i$ can depend on the bad event $\mathbb{A}_j$ if $\mathsf{Vbl}_i \cap \mathsf{Vbl}_j \neq \emptyset$
- Suppose each bad event $\mathbb{A}_i$ depends on at most d other bad events
- Suppose we show that, for each bad event $\mathbb{A}_i$, the probability of its occurrence $\mathbb{P}[\mathbb{A}_i] \leqslant p$
- If $\mathrm{e}p(d+1) \leqslant 1$, then

$$\mathbb{P}[\neg\mathbb{A}_1, \ldots \neg\mathbb{A}_n] \geqslant \left(1 - \frac{1}{d+1}\right)^n > 0$$

Generalized Lovász Local Lemma

- We design an experiment with independent random variables $\mathbb{X}_1, \ldots, \mathbb{X}_m$
- We define bad events $\mathbb{A}_1, \ldots, \mathbb{A}_n$
- Let D_i be the set of indices of bad events that $\mathbb{A}_i$ depends on
- Suppose we exhibit the existence of numbers $(x_1, \ldots, x_n)$ such that the following holds. For each i, we have:

$$\mathbb{P}\left[\mathbb{A}_i\right] \leqslant x_i \prod_{j \in D_i} (1 - x_j)$$

- Then the following holds.

$$\mathbb{P}\left[\neg\mathbb{A}_1, \ldots, \neg\mathbb{A}_n\right] \geqslant \prod_{i=1}^{n} (1 - x_i) > 0$$

Notes

- Prove Lovász Local Lemma using Generalized Lovász Local Lemma
- The numbers $(x_1, \ldots, x_n)$ are not probabilities that add up to 1. This is an incorrect intuition
- Prove the following corollary of the generalized Lovász Local Lemma

Corollary

If, for all i, we have $\sum_{j \in D_i} \mathbb{P}\left[\mathbb{A}_j\right] < 1/4$, then

$$\mathbb{P}\left[\neg\mathbb{A}_1, \ldots, \neg\mathbb{A}_n\right] \geqslant \prod_{i=1}^{n}(1 - 2\mathbb{P}\left[\mathbb{A}_i\right]) > 0$$

- Prove the results in the previous lecture using this corollary albeit with slightly worse parameters

Definition (Frugal Coloring)

A β-Frugal Coloring of a graph satisfies the following two conditions

1. It is a valid coloring, and
2. In the neighborhood $N(v)$ of any vertex v, there are at most β vertices with the same color.

For example, a 1-frugal coloring of G is a coloring of G^2

We will show the following result

Theorem

For $\beta \in \mathbb{N}$, and a graph G with maximum degree $\Delta \geqslant \beta^\beta$ there exists a β-frugal coloring using $16\Delta^{1+1/\beta}$ colors.

Note that a graph with maximum degree Δ can be 1-frugally colored with $\Delta^2 + 1$ colors. We will prove the general result using Lovász Local Lemma

Randomly color the vertices of the graph using C colors.
We will consider two types of bad events.

- $\mathbb{A}_e$, where $e \in E(G)$: If the two vertices at the endpoints of the edge e receive the same color. These will be called type-1 bad events.
- $\mathbb{B}_{\{u_1,\ldots,u_{\beta+1}\}}$, where $u_1, \ldots, u_{\beta+1} \in V(G)$: Suppose there exists a vertex v such that $u_1, \ldots, u_{\beta+1}$ are distinct vertices in $N(G)$ with identical color. These will be called type-2 bad events.

- Note that one type-1 bad event $\mathbb{A}_e$ can depends on at most 2Δ other type-1 bad events $\mathbb{A}_{e'}$
- We are now interested in computing how many type-2 bad events can $\mathbb{A}_e$ depend on. Consider a type-2 bad event $\mathbb{B}_{u_1,\ldots,u_{\beta+1}}$ such that that are in $u_1, \ldots, u_{\beta+1} \in N(v)$. Suppose $e = (a, b)$. Note that a has at most Δ neighbors. So, there are at most Δ possible ways of choosing v. Note that we have $\binom{\Delta}{\beta}$ ways of choosing the remaining vertices $\{u_1, \ldots, u_{\beta+1}\} \setminus \{a\}$. Similar case for b as well. So, there are at most $2\Delta\binom{\Delta}{\beta}$ type-2 events that $\mathbb{A}_e$ can depends on.

- Similarly, a type-2 event $\mathbb{B}_{u_1,\ldots,u_{\beta+1}}$ can depends on $(\beta+1)\Delta$ other type-1 bad events and $(\beta+1)\Delta\binom{\Delta}{\beta}$ other type-2 bad events
- Note that

$$\mathbb{P}\left[\mathbb{A}_e\right] \leqslant \frac{1}{C}$$
$$\mathbb{P}\left[\mathbb{B}_{u_1,\ldots,u_{\beta+1}}\right] \leqslant \frac{1}{C^\beta}$$

- So, to prove that a β-frugal coloring exists, it suffices to prove that

$$(\beta+1)\Delta\cdot\frac{1}{C}+(\beta+1)\Delta\binom{\Delta}{\beta}\cdot\frac{1}{C^\beta}<\frac{1}{4}$$

- We can use the upper bound $\binom{n}{k} \leqslant \left(\frac{\mathrm{e}n}{k}\right)^k$ to upper-bound the expression

$$\frac{(\beta+1)\Delta}{C} + \frac{(\beta+1)\Delta}{C^\beta}\binom{\Delta}{\beta}$$

- This is left as an exercise

Now we are interested in computing the solution that is guaranteed by Lovász Local Lemma.

function Seq_LLL($\mathbb{X} = \{\mathbb{X}_1, \ldots, \mathbb{X}_m\}, \mathcal{A} = \{\mathbb{A}_1, \ldots, \mathbb{A}_n\}$)
  $\mathbb{X} \leftarrow$ Random Evaluation
  while $\exists i$ s.t. $\mathbb{A}_i$ is satisfied **do**
    Pick arbitrary $\mathbb{A}_i$ that is satisfied
    Re-sample all $\mathbb{X}_j$ such that $j \in \mathrm{Vbl}_i$
  end while
  Output $\mathbb{X}$
end function

Performance of Moser-Tardos

Theorem

Suppose there exists $(x_1, \ldots, x_n)$ such that, for all $i \in [n]$, we have $\mathbb{P}[\mathbb{A}_i] \leqslant x_i \prod_{j \in D_i}(1 - x_j)$. Then the expected number of times sequential Moser-Tardos samples the event A_i is at most $x_i/(1 - x_i)$ and, hence, the expected number of execution of the inner loop is at most $\sum_{i \in [n]} x_i/(1 - x_i)$.

Parallel Moser-Tardos Algorithm

function Parallel_LLL($\mathbb{X}, \mathbb{A}$)
  $\mathbb{X} \leftarrow$ Random Evaluation
  while $\exists i$ s.t. $\mathbb{A}_i$ is satisfied **do**
    Let S be a maximal independent set in the dependency graph restricted to satisfied $\mathbb{A}_i$s
    $\mathbb{X}_{\cup_{k \in S} \text{Vbl}_k} \leftarrow$ Random Evaluation
  end while
  Output $\mathbb{X}$
end function

Performance of Parallel Moder-Tardos

Theorem

Suppose there exists an $\varepsilon > 0$ *and* $(x_1, \ldots, x_n)$ *such that* $\mathbb{P}[\mathbb{A}_i] \leqslant (1-\varepsilon)x_i \prod_{j \in D_i}(1-x_j)$. *The expected number of inner loops before all bad events are avoided is at most* $O\left(\frac{1}{\varepsilon}\sum_{i\in[n]}\frac{x_i}{1-x_i}\right)$.