Lecture 10: Lovász Local Lemma
Let $A_1, \ldots, A_n$ be indicator variables for bad events in an experiment.

Suppose $P[A_i] \leq p$

We want to avoid all the bad events.

If $P[\neg A_1 \land \cdots \land \neg A_n] > 0$, then there exists a way to avoid all the bad events simultaneously.

Suppose, the event $A_i$ is independent of all other events.

Then, it is easy to see that:

$$P[\neg A_1 \land \cdots \land \neg A_n] \geq (1 - p)^n > 0$$

Lovász Local Lemma will help us conclude the same even in presence of “limited independence”
The Statement

**Theorem (Lovász Local Lemma)**

Let \( (A_1, \ldots, A_n) \) be a set of bad event. For each \( A_i \), where \( i \in [n] \), we have \( P[A_i] \leq p \) and each event \( A_i \) depends on at most \( d \) other bad events. If \( ep(d + 1) \leq 1 \), then

\[
P[\neg A_1 \land \cdots \land \neg A_n] \geq \left(1 - \frac{1}{d + 1}\right)^n > 0
\]

The condition is also stated sometimes as \( 4pd \leq 1 \), instead of \( ep(d + 1) \leq 1 \).
Application: $k$-SAT

- Let $\Phi$ be a $k$-SAT formula such that each variable occurs in at most $2^{k-2}/k$ different clauses.
- Experiment: $X_i$ be an independent uniform random variable that assigns the variable $x_i$ a value from \{true, false\}.
- Bad Event: For the $j$-th clause we have the bad event $A_j$ that is the indicator variable for the bad event: The $j$-th clause is not satisfied.
- Probability of Bad Event: For any $j$, note that
  \[ \mathbb{P}[A_j] \leq \frac{1}{2^k}, \]
  because there is at most one assignment of variables to make the clause false.
Application: \( k \)-SAT

- **Dependence:** Note that the \( j \)-th clause has \( k \) literals, and each variable of the literal occurs in \( 2^{k-2}/k \) different clauses. So, the clause \( A_j \) can depend on at most \( d = 2^{k-2} \) different bad events.

- **Conclusion:** Note that \( 4pd = 1 \), so Lovász Local Lemma implies that there exists an assignment that satisfies all the clauses in the formula simultaneously.

- **Observation:** The probability \( p \) of each bad event does not depend on the overall problem instance size (i.e., the number of variables).
Application: Vertex Coloring

- Let $G$ be a graph with degree at most $\Delta$

- Experiment: $X_v$ be the random variable that represents the color of the vertex $v$. Let $X_v$ be an independent and uniformly random over the set $\{1, \ldots, C\}$

- Bad Event: For every edge $e$, we have a bad event $A_e$ that is the indicator variable for both its vertices receiving identical color

- Probability of the Bad Event: Note that $\mathbb{P}[A_e] = \frac{1}{C}$

- Dependence: Note that the event $A_e$ does not depend on any other event $A_{e'}$ if the edges do not share a vertex. So, the event $A_e$ depends on at most $2(\Delta - 1)$ other bad events

- Conclusion: A valid coloring exists if $4pd \leq 1$, i.e., $C \geq 8(\Delta - 1)$
Application: Vertex Coloring (Bad Bound)

- Let $G$ be a graph with degree at most $\Delta$
- Experiment: $X_v$ be the random variable that represents the color of the vertex $v$. Let $X_v$ be an independent and uniformly random over the set $\{1, \ldots, C\}$
- Bad Event: For every edge $v$, we have a bad event $A_v$ that is the indicator variable for one of the neighbors of $v$ receiving the same color as $v$
- Probability of the Bad Event: Note that $\Pr[A_v] \leq 1 - \left(1 - \frac{1}{C}\right)^\Delta$
- Dependence: Note that the event $A_v$ does not depend on any other event $A_{v'}$ if $\{v\} \cup N(v)$ does not intersect with $\{v'\} \cup N(v')$. So, the event $A_e$ depends on at most $\Delta + \Delta(\Delta - 1) = \Delta^2$ other bad events
- Conclusion: A valid coloring exists if $4pd \leq 1$, i.e., $C \geq ???$
Claim

Let $S \subseteq \{1, \ldots, n\}$, then we have:

$$
P \left[ A_i \mid \bigwedge_{k \in S} \neg A_k \right] \leq \frac{1}{d + 1}
$$

Assuming this claim, it is easy to prove the Lovász Local Lemma.

$$
P \left[ \bigwedge_{i=1}^{n} \neg A_i \right] = \prod_{i=1}^{n} P \left[ \neg A_i \mid \bigwedge_{k < i} \neg A_k \right]
\geq \prod_{i=1}^{n} \left( 1 - \frac{1}{d + 1} \right) = \left( 1 - \frac{1}{d + 1} \right)^n > 0
$$
Proof of the Claim

- We will proceed by induction on $|S|$
- **Base Case:** If $|S| = 0$, then the claim holds, because:

$$
P \left[ A_i \mid \bigwedge_{k \in S} \neg A_k \right] = P[A_i] \leq p \leq \frac{1}{e(d+1)} \leq \frac{1}{d+1}$$

- Assume that for all $S \mid S| < t$, the claim holds
- We will prove the claim for $|S| = t$. Suppose $D_i$ be the set of all $j$ such that the bad event $A_j$ depends on the bad event $A_i$
- **Easy Case.** Suppose $S \cap D_i = \emptyset$. This case is easy, because

$$
P \left[ A_i \mid \bigwedge_{k \in S} \neg A_k \right] = P[A_i] \leq p \leq \frac{1}{e(d+1)} \leq \frac{1}{d+1}$$

Lovász Local Lemma
Proof of the Claim

- **Remaining Case.** Suppose $S \cap D_i \neq \emptyset$.

\[
\Pr \left[ A_i \ \bigg| \ \bigwedge_{k \in S} \neg A_k \right] = \Pr \left[ A_i \ \bigg| \ \bigwedge_{k \in D_i} \neg A_k, \ \bigwedge_{k \in S \setminus D_i} \neg A_k \right]
\]

\[
= \frac{\Pr \left[ A_i, \ \bigwedge_{k \in D_i} \neg A_k \ \bigg| \ \bigwedge_{k \in S \setminus D_i} \neg A_k \right]}{\Pr \left[ \bigwedge_{k \in D_i} \neg A_k \ \bigg| \ \bigwedge_{k \in S \setminus D_i} \neg A_k \right]}
\]

\[
\times \frac{\Pr \left[ \bigwedge_{k \in S \setminus D_i} \neg A_k \right]}{\Pr \left[ A_i \ \bigg| \ \bigwedge_{k \in S \setminus D_i} \neg A_k \right]}
\]

\[
\leq \frac{\Pr \left[ A_i \right]}{\Pr \left[ \bigwedge_{k \in D_i} \neg A_k \ \bigg| \ \bigwedge_{k \in S \setminus D_i} \neg A_k \right]}
\]
Proof of the Claim

- Suppose $D_i = \{i_1, \ldots, i_z\}$
- Using chain rule, we can write the denominator as follows

\[
\prod_{\ell=1}^{z} \mathbb{P} \left[ \nabla_{k \in S \setminus D_i} \neg \bar{A}_k, \nabla_{k' \in \{i_1, \ldots, i_{\ell-1}\}} \neg \bar{A}_{k'} \right]
\]
Proof of the Claim

Note that each probability term is condition on $< t$ bad events. So, we can apply the induction hypothesis. We get

$$
P \left[ \bigwedge_{k \in D_i} \neg A_k \bigg| \bigwedge_{k \in S \setminus D_i} \neg A_k \right] \geq \prod_{\ell=1}^{z} \left( 1 - \frac{1}{d+1} \right)$$

$$= \left( 1 - \frac{1}{d+1} \right)^z \geq \left( 1 - \frac{1}{d+1} \right)^d$$

$$\geq \frac{1}{e}$$

Now, let us return to our original expression

$$P \left[ A_i \bigg| \bigwedge_{k \in S} \neg A_k \right] \leq \frac{P [A_i]}{P \left[ \bigwedge_{k \in D_i} \neg A_k \bigg| \bigwedge_{k \in S \setminus D_i} \neg A_k \right]}$$

$$\leq e P[A_i] \leq \frac{1}{d+1}$$
This completes the proof by induction.

We will prove a more general result in the next lecture.