Lecture 02: Mathematical Basics (Inequalities)
**Theorem (Basic AM-GM)**

For $a, b > 0$, we have

$$\frac{a + b}{2} \geq \sqrt{ab}$$

And equality holds if and only if $a = b$.

**Proof.**

$$\frac{a + b}{2} \geq \sqrt{ab} \iff \left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$$

The second statement is true for all reals. And, equality holds if and only if $a = b$. 

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**Mathematical Basics**
Theorem (Basic GM-HM)

For $a, b > 0$, we have

$$\sqrt{ab} \geq \left( \frac{\frac{1}{a} + \frac{1}{b}}{2} \right)^{-1}$$

And equality holds if and only if $a = b$.

Think: Proof? (it is a consequence of the Basic AM-GM Inequality)
First step is to note that

\[ \sum_{i=1}^{n} \frac{a_i}{n} \geq \left( \prod_{i=1}^{n} a_i \right)^{1/n} \]

This can be proven by induction on \( n \) and using the Basic AM-GM.
Generalizing AM-GM Inequality

Theorem ((Slight) Generalization of AM-GM)

For \( \alpha_1, \ldots, \alpha_n \in \mathbb{Q}^+ \) such that \( \sum_{i=1}^{n} \alpha_i = 1 \) and \( a_1, \ldots, a_n \geq 0 \), we have

\[
\sum_{i=1}^{n} \alpha_i a_i \geq \prod_{i=1}^{n} a_i^{\alpha_i}
\]

And, equality holds if and only if \( a_1 = \cdots = a_n \).

Think: Prove using Basic AM-GM. Let \( \alpha_i = p_i/q_i \) where \( p_i \) and \( q_i \) are relatively prime integers. Let \( N \) be the L.C.M. of \( \{q_1, \ldots, q_n\} \). Consider \( (p_i/q_i)N \) copies of \( a_i \), for \( i \in [n] \) and apply AM-GM on the \( N \) numbers.

Think: Generalize GM-HM analogously.

Further Generalization: Generalization to \( \alpha_i \in \mathbb{R} \) will be done later.
Theorem (Jensen’s Inequality)

Let $f$ be a convex downward function in the range $R$. Let $X$ be a probability distribution over $x_1, \ldots, x_n \in R$. We have

$$E[f(X)] \geq f(E[X])$$

Equality holds if and only if all $x_i$ are identical for $i$ such that $P[X = i] > 0$.

Clarification: “Convex Downward” function is a function that looks like the function $f(x) = x^2$ and does not look like the function $f(x) = \sqrt{x}$

Proof Intuition: Use induction on $n$. Base case of $n = 2$ is proven using: “the chord between two points lies above the function between the two points.”

Think: Analogous statement for convex upwards function
Jensen’s Inequality says that \( \frac{f(x_1) + f(x_2)}{2} \) is higher than \( f\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) \).
Let $f(x) = \log x$, for $x > 0$

Note that $f(x)$ is “convex upwards” (i.e., it looks like $\sqrt{x}$)

Let $X$ be the random variable over $[n]$ that outputs $i$ with probability $\alpha_i$

Let $x_i = a_i$ for $i \in [n]$

By Jensen’s Inequality we have $\mathbb{E} [f(X)] \leq f (\mathbb{E} [X])$

This equivalent to

$$\sum_{i \in [n]} \alpha_i \log x_i \leq \log \left( \sum_{i \in [n]} \alpha_i x_i \right)$$

Exponentiating both sides, we get the AM-GM inequality:

$$\prod_{i \in [n]} x_i^{\alpha_i} \leq \sum_{i \in [n]} \alpha_i x_i$$
Theorem (Generalized AM-GM)

For $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$ such that $\sum_{i=1}^{n} \alpha_i = 1$ and $a_1, \ldots, a_n \geq 0$, we have

$$\sum_{i=1}^{n} \alpha_i a_i \geq \prod_{i=1}^{n} a_i^{\alpha_i}$$

And, equality holds if and only if $a_1 = \cdots = a_n$. 
Cauchy–Schwarz Inequality

Theorem (Cauchy–Schwarz Inequality)

Let $a_1, \ldots, a_n, b_1, \ldots, b_n \geq 0$. Then the following holds

\[ \sum_{i=1}^{n} a_i b_i \leq \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} b_i^2 \right)^{1/2} \]

Equality holds if and only if $a_i/b_i$ is a constant for all $i \in [n]$.

Proof Outline:

- Prove the theorem for $n = 2$ using AM-GM inequality
- Prove for $n > 2$ using induction
Hölder’s inequality

**Theorem (Hölder’s inequality)**

Let $a_1, \ldots, a_n, b_1, \ldots, b_n \geq 0$. Let $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the following holds

$$\sum_{i=1}^{n} a_i b_i \leq \left( \sum_{i=1}^{n} a_i^p \right)^{1/p} \left( \sum_{i=1}^{n} b_i^q \right)^{1/q}$$

**Equality holds if and only if $a_i^p / b_i^q$ is a constant for all $i \in [n]$.**

**Proof Outline:**

- Assume the inequality holds for $n = 2$
- Use induction to extend the inequality to extend to $n > 2$
In this section we prove the full Hölder’s inequality in one-shot. The case of \( n = 2 \) is just a restriction of the analysis below to \( n = 2 \).

- Note that \( p, q > 0 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \) implies that \( p, q > 1 \)
- Consider the function \( f(x) = x^p \)
- For \( p > 1 \), this function is convex downwards
- Let \( x_i = a_i / b_i^{q/p} \)
- Let \( \alpha_i = \Lambda \cdot b_i^{1+q/p} \), where \( \Lambda \) is the normalizing constant such that \( \sum_{i \in [n]} \alpha_i = 1 \)
- By Jensen’s Inequality on \( f(x) = x^p \) we have:

\[
\sum_{i \in [n]} \alpha_i x_i^p \geq \left( \sum_{i \in [n]} \alpha_i x_i \right)^p
\]
Base Case of $n = 2$

Let be first find what is the value of $\Lambda$.

- $\sum_{k \in [n]} \alpha_k = \sum_{k \in [n]} \Lambda \cdot \frac{1}{p}^{1 + \frac{q}{p}} = 1$
- Note that $\frac{1}{p} + \frac{1}{q} = 1$. Multiplying both sides by $q$, we get $\frac{q}{p} + 1 = q$
- Now, we can substitute $\frac{q}{p} + 1 = q$ to get

$$\sum_{k \in [n]} \alpha_k = \Lambda \sum_{k \in [n]} b_k^q = 1$$

- This implies that $\Lambda = 1/ \sum_{k \in [n]} b_k^q$
Now, we can conclude that

\[ \alpha_i x_i = \frac{a_i b_i}{\sum_{k \in [n]} b_k^q}, \text{ and} \]

\[ \alpha_i x_i^p = \Lambda \cdot b_i^{1+\frac{q}{p}} \cdot \frac{a_i^p}{b_i^q} = \frac{a_i^p}{\sum_{k \in [n]} b_k^q} \]

Using the fact \( \frac{q}{p} + 1 = q \)

Now, let us substitute these values to simplify the equation we had obtained by applying the Jensen’s Inequality.

\[ \sum_{i \in [n]} \alpha_i x_i^p \geq \left( \sum_{i \in [n]} \alpha_i x_i \right)^p \]

\[ \iff \sum_{i \in [n]} \frac{a_i^p}{\sum_{k \in [n]} b_k^q} \geq \left( \sum_{i \in [n]} \frac{a_i b_i}{\sum_{k \in [n]} b_k^q} \right)^p \]

We continue this simplification in the next page
Base Case of $n = 2$

\[ \sum_{i \in [n]} \frac{a_i^p}{\sum_{k \in [n]} b_k^q} \geq \left( \sum_{i \in [n]} \frac{a_i b_i}{\sum_{k \in [n]} b_k^q} \right)^p \]

\[ \iff \left( \frac{a_i^p}{\sum_{k \in [n]} b_k^q} \right)^{1/p} \geq \sum_{i \in [n]} \frac{a_i b_i}{\sum_{k \in [n]} b_k^q} \]

\[ \iff \left( \sum_{k \in [n]} b_k^q \right)^{1-\frac{1}{p}} \left( \sum_{i \in [n]} a_i^p \right)^{1/p} \geq \sum_{i \in [n]} a_i b_i \]

\[ \iff \left( \sum_{k \in [n]} b_k^q \right)^{1/q} \left( \sum_{i \in [n]} a_i^p \right)^{1/p} \geq \sum_{i \in [n]} a_i b_i \]

\[ \therefore 1 - \frac{1}{p} = \frac{1}{q} \]

This completes the proof of the Hölder's Inequality
Taylor and Maclaurin Series

- Let $f$ be an infinitely differentiable function
- $f^{(n)}(x)$ represents $\frac{d^n f}{dx^n}(x)$. For $n = 0$, $f^{(n)}(x)$ represents $f(x)$
- Taylor Series of $f$ around $x_0$ is given by
  $$f(x) = \sum_{n \geq 0} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$
- Maclaurin Series of $f$ is the Taylor series with $x_0 = 0$
- Define the truncation of the Taylor series of $f$ up to $N$ terms as follows
  $$T_{f,N,x_0}(x) := \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$
- The remainder function is defined as follows
  $$R_{f,N,x_0}(x) := f(x) - T_{f,N,x_0}(x)$$
Theorem (The Remainder Theorem)

Suppose $f$ is $N + 1$ differentiable function. There exists $c$ between $x_0$ and $x$ such that

$$R_{f,N,x_0} = \frac{f^{(N+1)}(c)}{(N + 1)!} (x - x_0)^{N+1}$$

This theorem bounds the error between $f(x)$ and the truncation $T_{f,N,x_0}(x)$.
Application: Bounding $\exp(-x)$

- Let $f(x) = \exp(-x)$
- Note that $f^{(n)}(x) = (-1)^n \exp(-x)$
- Note that the Taylor series of $f(x)$ is

$$f(x) = \exp(-x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$$

- Note that $T_{f,1,0}(x) = 1 - x$ and $T_{f,2,0} = 1 - x + \frac{x^2}{2}$
- By applying the remainder theorem, we get

$$\exp(-x) - (1 - x) = R_{f,1,0} = \frac{\exp(-c)}{2!} x^2 \geq 0$$

$$\exp(-x) - (1 - x + \frac{x^2}{2}) = R_{f,2,0} = -\frac{\exp(-c')}{3!} x^3 \leq 0$$

- This implies that $1 - x \leq \exp(-x) \leq 1 - x + \frac{x^2}{2}$
Plots

\[ \exp(-x) \]

\[ 1 - x + \frac{x^2}{2} \]

Mathematical Basics