Lecture 25: Hypercontractivity and Applications

Hypercontractivity Theorem

Theorem

For $1 \leqslant p \leqslant q$ *and* $\rho = \sqrt{\frac{p-1}{q-1}}$ *the following is true:*

$$\|T_\rho(f)\|_q \leqslant \|f\|_p$$

Kahn-Kalai-Linial Theorem

- Let f be a function with output range $\{-1, 0, +1\}$
- Observe that $\|f\|_p^p = \Pr[f(x) \neq 0]$, for all $p > 0$

Theorem (Kahn-Kalai-Linial)

Let f be a function with range $\{-1, 0, +1\}$ and $\delta \in [0, 1]$. Then, the following holds:

$$\sum_{S \subseteq [n]} \delta^{|S|} \widehat{f}(S)^2 \leqslant \Pr[f(x) \neq 0]^{2/(1+\delta)}$$

- Let $q = 2$, $p = 1 + \delta \in [1, 2]$ and $\rho = \sqrt{p-1}$, then the Hypercontractivity Theorem gives us the following:

$$\Pr[f(x) \neq 0]^{2/(1+\delta)} = \|f\|_p^2 \geqslant \|T_\rho(f)\|_2^2 = \sum_{S \subseteq [n]} \delta^{|S|} \widehat{f}(S)^2$$

Application

- Let $A \subseteq \{0,1\}^n$
- We are interested in computing the parity of $\chi_S(x)$ when x is drawn randomly from A. So define the following bias:

$$\beta_S := \mathop{\mathbb{E}}_{x \xleftarrow{\$} A}[\chi_S(x)] = \frac{N}{|A|}\widehat{1_A}(S)$$

- We are interested in computing the average bias when S is a random k-subset of $[n]$. Towards that, let us compute the following:

$$\sum_{S \in \binom{[n]}{k}} \beta_S^2 = \frac{N^2}{|A|^2} \sum_{S \in \binom{[n]}{k}} \widehat{1_A}(S)^2 \leqslant \frac{N^2}{\delta^k |A|^2} \Pr[1_A(x) \neq 0]^{2/(1+\delta)}$$

$$= \frac{1}{\delta^k}\left(\frac{N}{|A|}\right)^{2\delta/(1+\delta)} \leqslant \frac{1}{\delta^k}\left(\frac{N}{|A|}\right)^{2\delta}$$

Application

- The right hand side is minimized for $\delta = k/2\ln(N/|A|)$
- So, we get:

$$\sum_{S \in \binom{[n]}{k}} \beta_S^2 \leqslant \left(\frac{2e\ln(N/|A|)}{k}\right)^k$$

- Therefore, we get:

$$\mathop{\mathbb{E}}_{S \xleftarrow{\$} \binom{[n]}{k}} [\beta_S^2] \leqslant \frac{\left(\frac{2e\ln(N/|A|)}{k}\right)^k}{\binom{n}{k}} \leqslant \left(\frac{2e\ln(N/|A|)}{n}\right)^k$$

$$= O\left(\frac{\ln(N/|A|)}{n}\right)^k$$

Intuition & Tightness

- For a *large* subset A, random parities of samples drawn from A are *nearly unbiased*
- Equivalently, A can be substituted by an arbitrary min-entropy distribution.

- To argue tightness consider $A = \{0,1\}^{n-c}0^c$
- There are $\binom{c}{k}$ subsets with bias 1, and rest have bias 0
- The expected bias-square is:

$$\frac{\binom{c}{k}}{\binom{n}{k}} \geqslant \left(\frac{c}{en}\right)^k = \Omega\left(\frac{\ln(N/|A|)}{n}\right)^k$$