Lecture 24: Hypercontractivity and Applications

Recall: Smoothening Operator

- Let $\varepsilon \in (0, 1/2]$
- Consider the following distribution $p(r) = \varepsilon^{\mathrm{wt}(r)}(1-\varepsilon)^{n-\mathrm{wt}(r)}$ (Intuition: Starting with input 0^n, each bit is independently flipped to 1 with probability ε to obtain the string r)
- Given a function f we define a smoothened version of f, namely, $\widetilde{f}$ as: $\widetilde{f}(x) := \sum_{r \in \{0,1\}^n} p(r) \cdot f(x+r)$

Recall: Properties

- Let $\rho = 1 - 2\varepsilon$
- Let $T_\rho(f) = \widetilde{f}$

Lemma

$$\widehat{\widetilde{f}}(S) = \rho^{|S|}\widehat{f}(S)$$

Lemma

The operator T_ρ is a linear bijection

Norm

Definition (p-Norm)

Given a function f, for $p > 0$, we define the p-norm as:

$$\|f\|_p := \left(\frac{1}{N} \sum_{x \in \{0,1\}^n} |f(x)|^p \right)^{1/p}$$

Norm: Properties

Lemma (Monotonicity)

For a fixed f, the $\|f\|_p$ is a non-decreasing function of p, for $p > 0$. The norm does not increase with increasing p if and only if f is a constant.

Lemma

For $p > 0$, suppose $\|f\|_p = \|g\|_p$ and let $h = \lambda \cdot f + (1 - \lambda) \cdot g$, where $\lambda \in [0, 1]$. Then $\|h\|_p \leqslant \|f\|_p$.

Norm of $T_\rho(\cdot)$

Lemma (Contraction)

For $p > 0$, we have:

$$\|T_\rho(f)\|_p \leqslant \|f\|_p$$

Lemma (Monotonicity)

For $\rho \leqslant \sigma$ and $p > 0$, we have:

$$\|T_\rho(f)\|_p \leqslant \|T_\sigma(f)\|_p$$

Theorem (Hypercontractivity)

For $1 \leqslant p \leqslant q$ and $\rho \leqslant \sqrt{\frac{p-1}{q-1}}$, we have:

$$\|T_\rho(f)\|_q \leqslant \|f\|_p$$

- Intuition: $\|T_\rho(f)\|_p$ is definitely $\leqslant \|f\|_p$. But this theorem says that even the $\|T_\rho(f)\|_q$ is $\leqslant \|f\|_p$.
- For $\rho = \sqrt{\frac{p-1}{q-1}}$ we get the tightest version of the inequality
- Application: We use $p = 2$ or $q = 2$, and then use Parseval's

Degree of Functions

Definition (Degree)

A function f has degree (at most) k if $\widehat{f}(S) = 0$ for all $|S| > k$.

Application: Bounds on Norms of Low-degree Polynomial

Let f be a function with degree k

Lemma

For $p \in [1,2]$, we have: $\|f\|_p^2 \geqslant (p-1)^k \|f\|_2^2$

Let $q = 2$ and $\rho = \sqrt{p-1}$ and $p \leqslant q$

$$
\begin{aligned}
\|f\|_p^2 &\geqslant \|T_\rho(f)\|_q^2 && \text{Hypercontractivity} \\
&= \sum_S \widehat{\widehat{f}}(S)^2 && \text{Parseval's} \\
&= \sum_S (p-1)^{|S|} \widehat{f}(S)^2 \geqslant (p-1)^k \sum_S \widehat{f}(S)^2 && \\
&= (p-1)^k \|f\|_2^2 && \text{Parseval's}
\end{aligned}
$$

- $\|f\|_p$ is definitely smaller than $\|f\|_2$ (because $p \leqslant 2$), but it is not *too small*

Application: Bounds on Norms of Low-degree Polynomial

Lemma

For $q \geqslant 2$, we have: $\|f\|_q^2 \leqslant (q-1)^k \|f\|_2^2$

Let $p = 2$, $\rho = 1/\sqrt{q-1}$ and g be such that $T_\rho(g) = f$

$$
\begin{aligned}
\|T_\rho(g)\|_q^2 &\leqslant \|g\|_2^2 && \text{Hypercontractivity} \\
&= \sum_S \widehat{g}(S)^2 && \text{Parseval's} \\
&= \sum_S \widehat{f}(S)^2/\rho^{2|S|} = \sum_S (q-1)^{|S|}\widehat{f}(S)^2 && \\
&\leqslant \sum_S (q-1)^k \widehat{f}(S)^2 && \text{Low Degree} \\
&\leqslant (q-1)^k \|f\|_2^2 && \text{Parseval's}
\end{aligned}
$$

- $\|f\|_q$ is definitely larger than $\|f\|_2$ (because $g \geqslant 2$), but it is not *too large*