Lecture 19: Fourier Analysis on the Boolean Hypercube

Functions

- We will deal with functions $f\colon \{0,1\}^n \to \mathbb{R}$
- Function f can be represented by a vector:

$$(f(0), f(1), \ldots, f(N-1)),$$

where $N = 2^n - 1$

- Any vector in $\mathbb{R}^N$ can be interpreted as a function

Definition (Inner Product)

Inner product of two functions $f, g\colon \{0,1\}^n \to \mathbb{R}$ is defined to be:

$$\langle f, g \rangle := \mathop{\mathbb{E}}_{x \sim U_n} [f(x)g(x)] = \frac{1}{N} \sum_{x=0}^{N-1} f(x)g(x)$$

Characters

Definition

For a subset $S \subseteq [n]$, we define the character function $\chi_S \colon \{0,1\}^n \to \mathbb{R}$ as follows:

$$\chi_S(x) = (-1)^{S \cdot x}$$

- We identify S with its characteristic vector $\in \{0,1\}^n$
- There are N such functions
- These N functions form an alternate basis to to express the space of all functions

Useful Observation

Lemma

For $A \subseteq [n]$, we have:

$$\sum_{x \in \{0,1\}^n} (-1)^{A \cdot x} = \begin{cases} N, & \text{if } A = \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

- If $A = \emptyset$, then $\sum_{x \in \{0,1\}^n} (-1)^{A \cdot x} = \sum_{x \in \{0,1\}^n} (-1)^0 = N$
- If $A \neq \emptyset$, then assume that $t \in A$ and $A' = A \setminus \{t\}$

$$\begin{aligned} \sum_{x \in \{0,1\}^n} (-1)^{A \cdot x} &= \sum_{x_1 \in \{0,1\}} \cdots \sum_{x_n \in \{0,1\}} (-1)^{A \cdot x} \\ &= \sum_{x_{[n] \setminus \{t\}} \in \{0,1\}^{n-1}} \sum_{x_t \in \{0,1\}} (-1)^{A \cdot x} \\ &= \sum_{x_{[n] \setminus \{t\}} \in \{0,1\}^{n-1}} (-1)^{A' \cdot x_{[n] \setminus \{t\}}} \sum_{x_t \in \{0,1\}} (-1)^{x_t} \end{aligned}$$

Proof continued

- Note that $\sum_{x_t \in \{0,1\}} (-1)^{x_t} = 0$
- So, we get

$$\sum_{x \in \{0,1\}^n} (-1)^{A \cdot x} = \sum_{x_{[n] \setminus \{t\}} \in \{0,1\}^{n-1}} (-1)^{A' \cdot x_{[n] \setminus \{t\}}} \cdot 0 = 0$$

Orthonormal Basis

Lemma

$\{\chi_S : S \subseteq [n]\}$ *is an orthonormal basis. In particular:*

$$\langle \chi_S, \chi_T \rangle = \begin{cases} 1, & \text{if } S = T \\ 0, & \text{otherwise.} \end{cases}$$

- Note that:

$$\langle \chi_S, \chi_T \rangle = \frac{1}{N} \sum_{x \in \{0,1\}^n} (-1)^{S \cdot x} \cdot (-1)^{T \cdot x} = \frac{1}{N} \sum_{x \in \{0,1\}^n} (-1)^{(S \Delta T) \cdot x}$$

- $S \Delta T = \emptyset$ if and only if $S = T$
- Using previous lemma, we get this result

Fourier Transform

Definition

Fourier Transform Given $f : \{0,1\}^n \to \mathbb{R}$, we define the following function:

$$\widehat{f} = \left(\widehat{f}(S=0), \widehat{f}(S=1), \ldots, \widehat{f}(S=N-1)\right),$$

where, for $S \subseteq [n]$, we define:

$$\widehat{f}(S) = \langle f, \chi_S \rangle$$

- Note that $\widehat{f}(S) = \frac{1}{N} \sum_{x \in \{0,1\}^n} f(x) \chi_S(x)$
- The Fourier transform $\mathcal{F}$ is a mapping that maps f to $\widehat{f}$
- And, we have $f = \sum_{S \subseteq [n]} \widehat{f}(S) \chi_S$

Linear & Bijective Map

Lemma

$f \mapsto_{\mathcal{F}} \widehat{f}$ is a linear bijective map.

- Consider the matrix $M \in \mathbb{R}^{N \times N}$ such that $M_{i,j} = \frac{1}{N}\chi_j(i)$
- Note that $\widehat{f}(j) = \sum_{i \in \{0,1\}^n} f(i) \cdot \frac{1}{N}\chi_j(i) = \sum_{i \in \{0,1\}^n} f(i) \cdot M_{i,j}$
- Therefore, $f \cdot M = \widehat{f}$
- This establishes that $\mathcal{F}$ is a linear map
- Note that M is a symmetric matrix and $M \cdot (N \cdot M) = I_{N \times N}$ (by orthonormality of the Fourier Basis)
- This establishes that $\mathcal{F}$ has an inverse and, hence, is a bijection

Properties and Examples

- $\widehat{(cf)} = c\widehat{f}$ (Follows from Linearity of $\mathcal{F}$)
- $\widehat{\left(\widehat{f}\right)} = \frac{1}{N} f$ (Follows from the fact that $M \cdot M = N \cdot I_{N\times N}$)
- Think: If $f(x) = g(x - c)$ then what is the relation between $\widehat{f}$ and $\widehat{g}$?

- Let $f(x) = 1$, for all x, then $\widehat{f}(S) = \begin{cases} 1, & \text{if } S = \emptyset \\ 0, & \text{otherwise.} \end{cases}$
- Let $f = U_n$, then $\widehat{f}(S) = \begin{cases} 1/N, & \text{if } S = \emptyset \\ 0, & \text{otherwise.} \end{cases}$
- Let $f = \delta_0$, then $\widehat{f}(S) = U_n$ (By linearity of $\mathcal{F}$ and the fact that $\mathcal{F}$ is its own (scaled) inverse)
- For any probability distribution f, we have $\widehat{f}(\emptyset) = \frac{1}{N}$

Example

Lemma

Let $V \subseteq \{0,1\}^n$ be a vector space of dimension t. Let $V^{\perp} \subseteq \{0,1\}^n$ be the orthogonal vector space of dimension $(n-t)$. Let $f = U_V$, that is f is a uniform distribution over V and 0 everywhere else. Then $\widehat{f}(S) = \begin{cases} \frac{1}{N}, & \text{if } S \in V^{\perp} \\ 0, & \text{otherwise.} \end{cases}$.

- Think about a proof.

Properties: Inner-product of Functions

Lemma

$$\langle f, g\rangle = \sum_{S\subseteq[n]} \widehat{f}(S)\widehat{g}(S)$$

- $f = \sum_S \widehat{f}(S)\chi_S$ and $g = \sum_T \widehat{g}(T)\chi_T$
- So, we have:

$$\begin{aligned}
\langle f, g\rangle &= \mathop{\mathbb{E}}_{x\sim U_n}\left[f(x)\cdot g(x)\right] \\
&= \mathop{\mathbb{E}}_{x\sim U_n}\left[\left(\sum_{S\subseteq[n]} \widehat{f}(S)\chi_S(x)\right)\cdot\left(\sum_{T\subseteq[n]} \widehat{g}(T)\chi_T(x)\right)\right] \\
&= \sum_{S\subseteq[n]}\sum_{T\subseteq[n]} \widehat{f}(S)\widehat{g}(T) \mathop{\mathbb{E}}_{x\sim U_n}\left[\chi_S(x)\cdot\chi_T(x)\right] \\
&= \sum_{S\subseteq[n]}\sum_{T\subseteq[n]} \widehat{f}(S)\widehat{g}(T)\mathbf{1}(S=T) = \sum_{S\subseteq[n]} \widehat{f}(S)\widehat{g}(S)
\end{aligned}$$

Parseval's Identity

- We define $\|f\|_2 = \sqrt{\langle f, f \rangle}$

Lemma (Parseval's Identity)

$$\|f\|_2^2 = \sum_{S \subseteq [n]} \widehat{f}(S)^2$$

- Follows from the inner product of two functions

Lemma

$$\mathrm{SD}\left(f, U_n\right) = \frac{N}{2}\left(\sum_{S\neq\emptyset}\widehat{f}(S)^2\right)^{1/2}$$

$$2\mathrm{SD}\left(f, U_n\right) = \sum_{x\in\{0,1\}^n}\left|f(x) - U_n(x)\right| = \sum_{x\in\{0,1\}^n}\left|(f - U_n)(x)\right|$$

$$\leqslant N\left\|f - U_n\right\|_2, \text{ By Chauchy-Schwartz}$$

$$= N\left(\sum_S \widehat{(f - U_n)}(S)^2\right)^{1/2} = N\left(\sum_S\left(\widehat{f}(S) - \widehat{U_n}(S)\right)^2\right)^{1/}$$

$$= N\left(\left(\widehat{f}(\emptyset) - \widehat{U_n}(\emptyset)\right)^2 + \sum_{S\neq\emptyset}\left(\widehat{f}(S) - \widehat{U_n}(S)\right)^2\right)^{1/2}$$

$$(\quad)^{1/2}$$

Bias

- Let $f_1 : \{0,1\} \to [0,1]$ be a probability distribution over one-bit
- $\mathrm{bias}(f_1) = 2\mathrm{SD}\left(f_1, U_1\right)$
- Equivalently: f_1 has bias α if and only if $f_1(b) \in \left\{\frac{1}{2} - \frac{\alpha}{2}, \frac{1}{2} + \frac{\alpha}{2}\right\}$, for $b \in \{0,1\}$

Definition (Bias)

Let f be a probability distribution over $\{0,1\}^n$ and $S \subseteq [n]$. Let f_S be a distribution over $\{0,1\}$ that outputs $\oplus_{i \in S} x_i$, when $x \sim f$. We define $\mathrm{bias}_S(f) = \mathrm{bias}(f_S)$.

- Think: $\mathrm{bias}_S(f) = N\left|\widehat{f}(S)\right|$