Lecture 15: Shearer's Lemma: Examples

- Let $\mathcal{F} \subseteq 2^{[n]}$
- For every $i \in [n]$, we have $|\{F : i \in F \in \mathcal{F}\}| \geqslant t$
- Shearer's Lemma:

$$H(X_1, \dots, X_n) \leqslant \frac{1}{t} \sum_{F \in \mathcal{F}} H(X_F)$$

Combinatorial Shearer's Lemma

- Let $\mathcal{A} \subseteq 2^{[n]}$
- Let $\mathrm{trace}_F(\mathcal{A}) = \{F \cap A \colon A \in \mathcal{A}\}$
- Combinatorial Shearer's Lemma:

$$|\mathcal{A}| \leqslant \left(\prod_{F \in \mathcal{F}} |\mathrm{trace}_F(\mathcal{A})| \right)^{1/t}$$

Independent Sets in Bipartite Graphs

- Let $i(H)$ be the number of independent sets of a graph H
- Let $G = (A, B, E)$ be a d-regular bipartite graph with $|A| = |B| = m$

Theorem (Kahn and Lawrenz)

$$i(G) \leqslant (2^{d+1} - 1)^{m/d}$$

- Tight when G is m/d copies of $K_{d,d}$ because $i(K_{d,d}) = (2^{d+1} - 1)$

Proof

- Let X be a uniformly chosen independent set of G
- X can be interpreted as $(X_v : v \in V(G))$, where X_v is 1 if $v \in X$, otherwise X_v is 0
- We can write $\log i(G) = H(X) = H(X_A | X_B) + H(B)$
- We can write:
  $H(X_A | X_B) \leqslant \sum_{v \in A} H(X_v | X_B) \leqslant \sum_{v \in A} H(X_v | X_{N(v)})$
- By Shearer's Lemma: $H(B) \leqslant \frac{1}{d} \sum_{v \in A} H(X_{N(v)})$
- Overall, we have:
  $\log i(G) \leqslant \frac{1}{d} \sum_{v \in A} dH(X_v | X_{N(v)}) + H(X_{N(v)})$
- Next, we analyze the term being summed

Proof (continued)

- Let $\overrightarrow{S}$ represent which vertices in $X_{N(v)}$ are included in X
- Let $e(\overrightarrow{S})$ represent the number of possible choices for X_v when $X_{N(v)} = \overrightarrow{S}$
- Note that if $\overrightarrow{S}$ is non-empty then $e(\overrightarrow{S}) = 1$, because $X_v = 0$ if $X_{N(v)} = \overrightarrow{S}$; if $\overrightarrow{S}$ is empty then $e(\overrightarrow{S}) = 2$, because $X_v = 0$ or $X_v = 1$ if $X_{N(v)} = \overrightarrow{S}$
- Note that: $H(X_{N(v)}) = \sum_{\overrightarrow{S}} p(\overrightarrow{S}) \log \frac{1}{p(\overrightarrow{S})}$
- Note that: $dH(X_v | X_{N(v)}) = \sum_{\overrightarrow{S}} dp(\overrightarrow{S}) H(X_v | X_{N(v)} = \overrightarrow{S}) \leqslant \sum_{\overrightarrow{S}} p(\overrightarrow{S}) \log e(\overrightarrow{S})^d$
- So, we have: $dH(X_v | X_{N(v)}) + H(X_{N(v)}) = \sum_{\overrightarrow{S}} p(\overrightarrow{S}) \log \frac{e(\overrightarrow{S})^d}{p(\overrightarrow{S})} \leqslant_{(*)} \log \sum_{\overrightarrow{S}} e(\overrightarrow{S})^d = \log i(K_{d,d})$, where $(*)$ is due to Jensen's Inequality
- Overall, we get: $\log i(G) \leqslant \frac{1}{d} \sum_{v \in A} \log i(K_{d,d}) = \log i(K_{d,d})^{m/d}$

q-Colorings of Bipartite Graphs

- Let $c_q(H)$ be the number of q-colorings of a graph H
- Let $G = (A, B, E)$ be a d-regular bipartite graph with $|A| = |B| = m$

Theorem (Galvin and Tetali)

$$c_q(G) \leqslant c_q(K_{d,d})^{m/d}$$

- Note that the inequality is tight when G is m/d copies of $K_{d,d}$

Proof

- Let X be a uniformly chosen q-coloring of G
- Let $X \equiv (X_v : v \in V(G))$ where X_v corresponds to the color of vertex v
- So, we have $\log c_q(G) = H(X) = H(X_A|X_B) + H(X_B)$
- Note that:
  $H(X_A|X_B) \leqslant \sum_{v \in A} H(X_v|X_B) \leqslant \sum_{v \in A} H(X_v|X_{N(v)})$
- By Shearer's Lemma: $H(X_B) \leqslant \frac{1}{d} \sum_{v \in A} H(X_{N(v)})$
- So, overall we have:
  $\log c_q(G) \leqslant \frac{1}{d} \sum_{v \in A} dH(X_v|X_{N(v)}) + H(X_{N(v)})$
- Next, we analyze the term being summed

Proof (continued)

- Let $\overrightarrow{C}$ be a q-coloring of $N(v)$
- Then, we have: $H(X_{N(v)}) = \sum_{\overrightarrow{C}} p(\overrightarrow{C}) \log \frac{1}{p(\overrightarrow{C})}$
- Let $e(\overrightarrow{C})$ be the number of different possible colorings of v given $\overrightarrow{C}$
- Then, we have: $dH(X_v|X_{N(v)}) = \sum_{\overrightarrow{C}} dp(\overrightarrow{C}) H(X_v|X_{N(v)} = \overrightarrow{C}) \leqslant \sum_{\overrightarrow{C}} p(\overrightarrow{C}) \log e(\overrightarrow{C})^d$
- Summing: $dH(X_v|X_{N(v)}) + H(X_{N(v)}) \leqslant \sum_{\overrightarrow{C}} p(\overrightarrow{C}) \log \frac{e(\overrightarrow{C})^d}{p(\overrightarrow{C})} \leqslant_{(*)} \log \sum_{\overrightarrow{C}} e(\overrightarrow{C})^d = \log c_q(K_{d,d})$, where $(*)$ is by Jensen's Inequality
- Overall, we get the upper bound: $\log c_q(G) \leqslant \frac{1}{d} \sum_{v \in A} \log c_q(K_{d,d}) = \log c_q(K_{d,d})^{m/d}$

Beyond

- Is this symptomatic of a more general phenomenon?