Lecture 15: Shearer’s Lemma: Examples
Shearer’s Lemma

- Let $\mathcal{F} \subseteq 2^{[n]}$
- For every $i \in [n]$, we have $|\{F : i \in F \in \mathcal{F}\}| \geq t$
- Shearer’s Lemma:

$$H(X_1, \ldots, X_n) \leq \frac{1}{t} \sum_{F \in \mathcal{F}} H(X_F)$$
Let $\mathcal{A} \subseteq 2^{[n]}$.

Let $\text{trace}_F(\mathcal{A}) = \{F \cap A : A \in \mathcal{A}\}$.

Combinatorial Shearer’s Lemma:

$$|\mathcal{A}| \leq \left( \prod_{F \in \mathcal{F}} |\text{trace}_F(\mathcal{A})| \right)^{1/t}$$
Let \( i(H) \) be the number of independent sets of a graph \( H \)

Let \( G = (A, B, E) \) be a \( d \)-regular bipartite graph with \( |A| = |B| = m \)

**Theorem (Kahn and Lawrenz)**

\[
i(G) \leq (2^{d+1} - 1)^{m/d}
\]

Tight when \( G \) is \( m/d \) copies of \( K_{d,d} \) because

\[
i(K_{d,d}) = (2^{d+1} - 1)
\]
Proof

- Let $X$ be a uniformly chosen independent set of $G$

  - $X$ can be interpreted as $(X_v : v \in V(G))$, where $X_v$ is 1 if $v \in X$, otherwise $X_v$ is 0

- We can write $\log i(G) = H(X) = H(X_A | X_B) + H(B)$

- We can write:
  \[ H(X_A | X_B) \leq \sum_{v \in A} H(X_v | X_B) \leq \sum_{v \in A} H(X_v | X_{N(v)}) \]

- By Shearer’s Lemma: $H(B) \leq \frac{1}{d} \sum_{v \in A} H(X_{N(v)})$

- Overall, we have:
  \[ \log i(G) \leq \frac{1}{d} \sum_{v \in A} dH(X_v | X_{N(v)}) + H(X_{N(v)}) \]

- Next, we analyze the term being summed
Proof (continued)

- Let $\vec{S}$ represent which vertices in $X_{N(v)}$ are included in $X$
- Let $e(\vec{S})$ represent the number of possible choices for $X_v$ when $X_{N(v)} = \vec{S}$
- Note that if $\vec{S}$ is non-empty then $e(\vec{S}) = 1$, because $X_v = 0$ if $X_{N(v)} = \vec{S}$; if $\vec{S}$ is empty then $e(\vec{S}) = 2$, because $X_v = 0$ or $X_v = 1$ if $X_{N(v)} = \vec{S}$
- Note that: 
  \[
  H(X_{N(v)}) = \sum_{\vec{S}} p(\vec{S}) \log \frac{1}{p(\vec{S})}
  \]
- Note that: 
  \[
  dH(X_v | X_{N(v)}) = \sum_{\vec{S}} dp(\vec{S}) H(X_v | X_{N(v)} = \vec{S}) \leq \sum_{\vec{S}} p(\vec{S}) \log e(\vec{S})^d
  \]
- So, we have: 
  \[
  dH(X_v | X_{N(v)}) + H(X_{N(v)}) = \sum_{\vec{S}} p(\vec{S}) \log \frac{e(\vec{S})^d}{p(\vec{S})} \leq (*) \log \sum_{\vec{S}} e(\vec{S})^d = \log i(K_d,d), \text{ where} \]
  \[
  (*) \text{ is due to Jensen's Inequality}
  \]
- Overall, we get: 
  \[
  \log i(G) \leq \frac{1}{d} \sum_{v \in A} \log i(K_d,d) = \log i(K_d,d)^{m/d}
  \]
Let $c_q(H)$ be the number of $q$-colorings of a graph $H$

Let $G = (A, B, E)$ be a $d$-regular bipartite graph with $|A| = |B| = m$

Theorem (Galvin and Tetali)

$$c_q(G) \leq c_q(K_{d,d})^{m/d}$$

Note that the inequality is tight when $G$ is $m/d$ copies of $K_{d,d}$
Proof

- Let $X$ be a uniformly chosen $q$-coloring of $G$
- Let $X \equiv (X_v : v \in V(G))$ where $X_v$ corresponds to the color of vertex $v$
- So, we have $\log c_q(G) = H(X) = H(X_A|X_B) + H(X_B)$
- Note that:
  $H(X_A|X_B) \leq \sum_{v \in A} H(X_v|X_B) \leq \sum_{v \in A} H(X_v|X_{N(v)})$
- By Shearer’s Lemma: $H(X_B) \leq \frac{1}{d} \sum_{v \in A} H(X_{N(v)})$
- So, overall we have:
  $\log c_q(G) \leq \frac{1}{d} \sum_{v \in A} dH(X_v|X_{N(v)}) + H(X_{N(v)})$
- Next, we analyze the term being summed
Proof (continued)

- Let $\vec{C}$ be a $q$-coloring of $N(v)$
- Then, we have: $H(X_{N(v)}) = \sum_{\vec{C}} p(\vec{C}) \log \frac{1}{p(\vec{C})}$
- Let $e(\vec{C})$ be the number of different possible colorings of $v$ given $\vec{C}$
- Then, we have: $dH(X_v|X_{N(v)}) = \sum_{\vec{C}} dp(\vec{C}) H(X_v|X_{N(v)} = \vec{C}) \leq \sum_{\vec{C}} p(\vec{C}) \log e(\vec{C})^d$
- Summing: $dH(X_v|X_{N(v)}) + H(X_{N(v)}) \leq \sum_{\vec{C}} p(\vec{C}) \log \frac{e(\vec{C})^d}{p(\vec{C})} \leq(*) \log \sum_{\vec{C}} e(\vec{C})^d = \log c_q(K_{d,d})$
  where $(\ast)$ is by Jensen’s Inequality
- Overall, we get the upper bound:
  $\log c_q(G) \leq \frac{1}{d} \sum_{v \in A} \log c_q(K_{d,d}) = \log c_q(K_{d,d})^{m/d}$
Is this symptomatic of a more general phenomenon?