Lecture 14: Shearer's Lemma: Examples

Shearer's Lemma

- Let $\mathcal{F} \subseteq 2^{[n]}$
- For every $i \in [n]$, we have $|\{F : i \in F \in \mathcal{F}\}| \geqslant t$
- Shearer's Lemma:

$$H(X_1, \ldots, X_n) \leqslant \frac{1}{t} \sum_{F \in \mathcal{F}} H(X_F)$$

Combinatorial Shearer's Lemma

- Let $\mathcal{A} \subseteq 2^{[n]}$
- Let $\mathrm{trace}_F(\mathcal{A}) = \{F \cap A \colon A \in \mathcal{A}\}$
- Combinatorial Shearer's Lemma:

$$|\mathcal{A}| \leqslant \left(\prod_{F \in \mathcal{F}} |\mathrm{trace}_F(\mathcal{A})| \right)^{1/t}$$

Volume

- Let S be a set of n points
- Let S_i be the set of n_i points where the i-th coordinate of elements in S is dropped, for $i \in [d]$

Theorem

$$n \leqslant (n_1 \cdots n_d)^{1/(d-1)}$$

- Let $(X_1, \ldots, X_d)$ be random variable for the coordinates of a uniformly random point in S
- Let $F_{-i} = [n] \setminus \{i\}$
- note that $\left|\text{range}(X_{F_{-i}})\right| = n_i$
- Let $\mathcal{F} = \{F_{-1}, \ldots, F_{-d}\}$
- Note that each element $i \in [n]$ is contained in at least $(d-1)$ sets in $\mathcal{F}$
- By Shearer's Lemma:
  $\log n = H(X) \leqslant \frac{1}{d-1} \sum_{F_{-i} \in \mathcal{F}} H(X_{F_{-i}}) \leqslant \frac{1}{d-1} \sum_{F_{-i} \in \mathcal{F}} \log n_i$

Intersecting Family of Graphs

Theorem

Let $\mathcal{G}$ be a family of graphs over n vertices such that for all $G_i, G_j \in \mathcal{G}$, there exists a triangle in $G_i \cap G_j$. Then $|\mathcal{G}| \leqslant 2^m - 2$, where $m = \binom{n}{2}$.

- It is clear that there exists $\mathcal{G}$ such that $|\mathcal{G}| = 2^{m-3}$ (because, one can fix a triangle and every other edge we get an option to pick it or not)
- It is clear that $|\mathcal{G}| \leqslant 2^{m-1}$, because no set and its complement can be included
- Make the bound to $|\mathcal{G}| \leqslant 2^{m-2}$

Intersecting Family of Graphs (continued)

- Let (A, B) be a partition of n vertices such that $|A| = \lfloor n/2 \rfloor$ and $|B| = \lceil n/2 \rceil$
- Let $U(A, B)$ be the graph that is complete over all vertices in A and complete over all vertices in B
- Note that $G_i \cap G_j$ contains a triangle. So, $(G_i \cap G_j) \cap U(A, B) \neq \emptyset$ (because, if this is empty then $(G_i \cap G_j)$ is a bipartite graph)
- Therefore $(G_i \cap U(A, B)) \cap (G_j \cap U(A, B))$ is non-empty
- So, any two graphs in $\mathrm{trace}_{U(A,B)}(\mathcal{G})$ intersects
- Therefore, $\mathrm{trace}_{U(A,B)}(\mathcal{G}) \leqslant 2^{m'-1}$, where m' is the number of edges in $U(A, B)$, i.e., $m' = \binom{\lceil n/2 \rceil}{2} + \binom{\lfloor n/2 \rfloor}{2}$
- Let $\mathcal{F}$ be the set of all $U(A, B)$ for all partitions (A, B)

Intersecting Family of Graphs (continued)

- Note that $m' |\mathcal{F}| = mt$, where any edge over the n vertices occurs t times
- Combinatorial Shearer's Lemma:

$$|\mathcal{G}| = \left(\prod_{F \in \mathcal{F}} |\mathrm{trace}_F(\mathcal{G})| \right)^{1/t} \leqslant \left(2^{m'-1} \right)^{|\mathcal{F}|/t} \leqslant 2^{m-2}$$

Counting Independent Sets in Bi-Partite Graphs

- Let $G = (A, B, E)$ be a d-regular bipartite graph with partite sets A and B, and edge set E
- Let $|A| = |B| = m$

Theorem

The number of independent sets N is at most $\frac{(}{2}^{d+1} - 1)^{m/d}$

- Let X be uniformly random independent set in G
- $\log N = H(X) = H(X_A|X_B) + H(X_B) \leqslant \left(\sum_{v\in A} H(X_v|X_B)\right) + H(X_B)$
- By Shearer's Lemma: $H(X_B) \leqslant \frac{1}{d}\sum_{v\in A} H(X_{N(v)})$
- $\log N \leqslant \sum_{v\in A} H(X_v|X_B) + \frac{1}{d}H(X_{N(v)}) \leqslant \sum_{v\in A} H(X_v|X_{N(v)}) + \frac{1}{d}H(X_{N(v)})$

Counting Independent Sets (continued)

- Let $\chi_v = 0$ if $X_{N(v)} = \mathbf{0}$, $\chi_v = 1$, otherwise
- Let $p = \Pr[\chi_v = 0]$
- First, $H(X_v | X_{N(v)}) \leqslant H(X_v | \chi_v) \leqslant 1 \cdot p + 0 \cdot (1-p) = p$
- Second, $H(X_{N(v)}) = H(X_{N(v)}, \chi_v) = H(\chi_v) + H(X_{N(v)} | \chi_v) \leqslant h(p) + (1-p)\log(2^d - 1)$
- So, we have $\log N \leqslant \sum_{v \in A} p + \frac{1}{d}\left(h(p) + (1-p)\log(2^d - 1)\right)$
- The right had side is maximized for $p = 2^d/(2^{d+1} - 1)$ and, hence, we get: $\log N \leqslant \frac{m}{d}\log(2^{d+1} - 1)$

Counting Embedding of Graphs

- Let H be a small graph and G be a graph with ℓ edges
- Let $\text{embed}(H, \ell)$ represent the number of embeddings of H in a graph with ℓ edges

Theorem

There exists $c_1, c_2, \rho^*(H)$ such that:

$$c_1 \ell^{\rho^*(H)} \leqslant \text{embed}(H, \ell) \leqslant c_2 \ell^{\rho^*(H)}$$

Example

- Let H be a triangle
- Let G be any graph with ℓ edges
- The number of embeddings of H in G such that one of the vertices of H falls at v is at most $d(v)^2$ (because the remaining two vertices of H can be mapped to two neighbors of v)
- The number of embeddings of H in G such that one of the vertices of H falls at v is at most 2ℓ (because the edge in H opposite to v can map to one of the ℓ different edges or their reverses)
- So, the number of embeddings is upper bounded by $\sum_{v \in V(G)} \min\{d_v^2, 2\ell\} \leqslant \sum_{v \in V(G)} \sqrt{d_v^2 \ell} = (2\ell)^{3/2}$
- Let G be a $K_{n \times n \times n}$ graph such that $3n^2 = \ell$, then there are $n^3 = (\ell/3)^{3/2}$ different mappings of H in G

Fractional Covering

- Covering:
  - Let $\varphi\colon E(H) \to \{0,1\}$
  - For every vertex $v \in V(H)$, we have: $\sum_{e \in E(H)\colon v \in e} \varphi(e) \geqslant 1$
  - Overall, we define: $\rho(H) = \min_{\varphi} \sum_{e \in E(H)} \varphi(e)$

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Fractional Independent Set

- Independent Set:
  - Let $\psi \colon V(H) \to \{0,1\}$
  - For every edge $e \in E(H)$, we have: $\sum_{v \in V(H)\colon v \in e} \psi(v) \leqslant 1$
  - Overall, we define $\alpha(H) = \max_{\psi} \sum_{v \in V(H)} \psi(v)$

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LP Duality

Theorem

$$\rho^*(H) = \alpha^*(H) \in \mathbb{Q}$$

- $\rho(H) \geqslant \rho^*(H) = \alpha^*(H) \geqslant \alpha(H)$

Lower Bound

- Consider the mapping ψ^* that achieves $\alpha^*(H)$
- We construct a graph G as follows:
  - For $v \in V(H)$, we construct partite set V_v of $(\ell / |E(H)|)^{\psi^*(v)}$
  - For every $(u, v) \in E(H)$ we connect every vertex in V_u to every vertex in V_v
- Total number of edges is:
  $\sum_{(u,v) \in E(H)} \left(\frac{\ell}{|E(H)|} \right)^{\psi^*(u) + \psi^*(v)} \leqslant \sum_{(u,v) \in E(H)} \frac{\ell}{|E(H)|} = \ell$
- $\text{embed}(H, \ell) \geqslant \prod_{v \in V(H)} \left(\frac{\ell}{|E(H)|} \right)^{\psi^*(v)} = c_1 \ell^{\alpha^*(H)} = c_1 \ell^{\rho^*(H)}$

Upper Bound

- $X_{V(H)}$ is uniformly chosen embedding
- Consider the mapping φ^* that achieves $\rho^*(H)$
- Let $C \in \mathbb{N}$ such that $C \cdot \varphi^*(e) \in \mathbb{N}$ for all $e \in E(H)$
- For every $(u, v) = e \in E(H)$ we consider the event $X_{u,v}$ with $C \cdot \varphi^*(e)$ repetitions in $\mathcal{F}$
- Every vertex v occurs in at least $\sum_{e \in E(H):\ v \in e} C\varphi^*(v) \geqslant C$ times
- $\log|\text{embed}(H, \ell)| \leqslant \frac{1}{C} \sum_{(u,v)\in E(H)} C\varphi^*(e) H(X_{\{u,v\}}) \leqslant \sum_{(u,v)\in E(H)} \varphi^*(v) \log(2\ell) = \rho^*(H) \log(2\ell)$
- $|\text{embed}(H, \ell)| \leqslant c_2 \ell^{\rho^*(H)}$