Lecture 11: Properties of Expanders and Graph Products

Notation: Recall

Let G be an undirected d-regular non-bipartite graph

- Let A be the adjacency matrix of G
- Let $M = \frac{1}{d} \cdot A$ be the normalized adjacency matrix of G
- Let $1 = \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n > -1$ be the eigenvalues of M
- Let $\lambda(G) = \max_{1 \leqslant i \leqslant n} |\lambda_i|$
- Let J be a matrix of all 1s
- Now,

$$\left\| M - \frac{1}{n} J \right\| = \max_{x : \|x\| = 1} \left\| \left(M - \frac{1}{n} J \right) x \right\| = \lambda(G)$$

Expander Mixing Lemma

Lemma (Expander Mixing Lemma)

Let $S, T \subset V$ be two disjoint subset of vertices. Then,

$$\left| E(S,T) - \frac{d}{|V|} \cdot |S| \cdot |T| \right| \leqslant \lambda(G) \cdot d \cdot \sqrt{|S| \cdot |T|}$$

- $E(S,T) = d \cdot 1_S^{\intercal} M 1_T$
- $|S| \cdot |T| = 1_S^{\intercal} J 1_T$
- Now, we have:

$$\begin{aligned}
\left| E(S,T) - \frac{d}{|V|} \cdot |S| \cdot |T| \right| &= d \cdot \left| 1_S^{\intercal} M 1_T - \frac{1}{|V|} 1_S^{\intercal} J 1_T \right| \\
&= d \cdot \left| 1_S^{\intercal} \left(M - \frac{1}{|V|} J \right) 1_T \right| \\
&\leqslant d \left\| 1_S \right\| \cdot \left\| M - \frac{1}{|V|} J \right\| \cdot \left\| 1_T \right\| \\
&= d \sqrt{|S|} \cdot \lambda(G) \cdot \sqrt{|T|}
\end{aligned}$$

Random Walks

A t-step random walk starting with distribution p is given by $M^t p$

Lemma (Mixing Time of Random Walks in Expanders)

Let p is be any starting probability distribution. Then,

$$\left\|u - M^t p\right\|_1 \leqslant \sqrt{V}(\lambda(G))^t$$

$$\begin{aligned}\left\|u - M^t p\right\|_1 &\leqslant \sqrt{|V|} \cdot \left\|u - M^t p\right\| = \sqrt{|V|} \cdot \left\|\frac{1}{|V|} Jp - M^t p\right\| \\ &\leqslant \sqrt{|V|}(\lambda(G))^t \|p\| \\ &\leqslant \sqrt{|V|}(\lambda(G))^t\end{aligned}$$

Diameter of Expanders

Think: Use previous result to prove a logarithmic bound on the diameter of an expander graph

Graph Products

Notation about Graphs:

- Given two regular undirected graphs G and H we will study different ways to combine them
- We will assume that every edge incident on a vertex v is named uniquely
- So, any edge (u, v) will receive two names i and j, where i corresponds to the vertex u and j corresponds to the vertex v
- This naming of edges can be arbitrary

Replacement Product

- Let G be a "large" D-regular graph on N vertices
- Let H be a "small" d-regular graph on D vertices
- Assume that in G, for any vertex v, the edges incident on v have an ordering
- Vertex set of $G \circledR H$ is $V(G) \times V(H)$
- (u, i) is connected to (v, j) if and only if:
  - $u = v$ and $(i, j) \in E(H)$, or
  - $u \neq v$, the edge $e = (u, v) \in E(G)$, and e is the i-th neighbor of u and j-th neighbor of v
- The graph $G \circledR H$ has ND vertices and is $(d+1)$-regular

Theorem (Expansion of Replacement Product Graph)

Let G be an (N, D, Λ) graph and H be a (D, d, λ) graph. Then, $G \circledR H$ is an $(ND, d+1, g(\Lambda, \lambda, d))$ graph, where:

$$g(\Lambda, \lambda, d) \leqslant (p + (1-p) f(\Lambda, \lambda))^{1/3}$$

and $p = d^2/(d+1)^3$ and $f(\Lambda, \lambda) \leqslant \Lambda + \lambda + \lambda^2$.

Zig-Zag Product

- (u, i) is connected to (v, j) if there exists k and ℓ such that (u, i) is connected to (u, k) is connected to (v, ℓ) is connected to (v, j) in $G \textcircled{r} H$

Theorem (Expansion of Replacement Product Graph)

Let G be an (N, D, Λ) graph and H be a (D, d, λ) graph. Then, $G \textcircled{z} H$ is an $(ND, d^2, f(\Lambda, \lambda, d))$ graph, where:

$$f(\Lambda, \lambda) \leqslant \Lambda + \lambda + \lambda^2$$

- $\text{Rot}(u, i) = (v, j)$, if there exists an edge (u, v) that is labeled i at the vertex u and labeled j at the vertex v
- Think: Compute the Rot mapping of $G \textcircled{r} H$ and $G \textcircled{z} H$ graphs given oracle access to Rot mapping of G and H graphs respectively