Lecture 09: Spectral Graph Theory
Let $G = (V, E)$ be an undirected graph

**Definition (Sparsity of a Cut)**

$$
\sigma(S) := \frac{\mathbb{E}_{(u,v) \in E} [1_S(u) - 1_S(v)]}{\mathbb{E}_{(u,v) \in V^2} [1_S(u) - 1_S(v)]}
$$

**Definition (Sparsity of a Graph)**

$$
\sigma(G) := \min_{S \subseteq V: S \neq \emptyset, S \neq V} \sigma(S)
$$

For a $d$-regular graph $G$, $\sigma(S) = \frac{E(S, V-S)}{d|S||V-S|/|V|}$
Let $G$ be a $d$-regular undirected graph.

**Definition (Edge Expansion of a Set)**

\[ \phi(S) := \frac{E(S, V - S)}{d|S|} \]

**Definition (Edge Expansion of a Graph)**

\[ \phi(G) := \min_{S: |S| \leq |V|/2} \phi(S) \]
Lemma

For a regular graph $G$,

\[ \frac{1}{2} \sigma(G) \leq \phi(G) \leq \sigma(G) \]

Proof is trivial
Definition (Family of Expander Graphs)

Let \( \{G_n\}_{n > d} \) be a family of \( d \)-regular graphs such that \( \phi(G_n) \geq \phi \). This family is called a family of expander graphs.
Lemma

Let $\phi(G) \geq \phi > 0$. Consider any $0 < \varepsilon < \phi$. On removal of any $\varepsilon |E|$ edges from $G$, there exists a connected component of $G$ of size at least $(1 - \varepsilon/2\phi) |V|$.

- Let $E'$ be any set of at most $\varepsilon |E|$ edges in $G$
- Let $C_1, \ldots, C_t$ be the connected components of $G'$ (the graph obtained from $G$ by removal of edges $E'$) in non-decreasing order
- If $|C_1| \leq |V|/2$:
  \begin{align*}
  |E'| & \geq \frac{1}{2} \sum_{i \neq j} E(C_i, C_j) = \frac{1}{2} \sum_i E(C_i, V - C_i) \\
  & \geq \frac{1}{2} \sum_i |C_i| \phi = \frac{d |V| \phi}{2}
  \end{align*}

This is impossible.
• If $|C_1| > |V|/2$:

$$|E'| \geq \mathbb{E}(C_1, V - C_1) \geq d\phi |V - C_1|$$

$$\implies |V - C_1| \leq \frac{\varepsilon d |V|}{2d\phi}$$

Hence, $|C_1| \geq (1 - \varepsilon/2\phi) |V|$
Hermitian Matrices

\[ \langle x, y \rangle := x^* y = \sum_i \bar{x}_i y_i \]
\[ \langle x, x \rangle = \|x\|^2 \]

A Hermitian matrix \( M \in \mathbb{C}^{n \times n} \) satisfies \( M = M^* \)

If \( Mx = \lambda x \) then \( \lambda \) is the eigenvalue and \( x \) is the corresponding eigenvector
Lemma

All eigenvalues of a Hermitian $M$ are real

- Suppose $\lambda$ is an eigenvalue and $x$ is its corresponding eigenvector.
- Consider $\langle Mx, x \rangle = \langle x, M^*x \rangle = \langle x, Mx \rangle$.
- Note that $\langle Mx, x \rangle = \overline{\lambda} \langle x, x \rangle$ and $\langle x, Mx \rangle = \lambda \langle x, x \rangle$.
- Hence, $\overline{\lambda} = \lambda$. 
Lemma

Let $x$ and $y$ be eigenvectors of a Hermitian $M$ corresponding to two different eigenvalues. Then, $\langle x, y \rangle = 0$.

- Let $\lambda$ and $\lambda'$ be eigenvalues corresponding to $x$ and $y$ respectively.
- Note that $\langle Mx, y \rangle = \lambda \langle x, y \rangle$.
- Note that $\langle x, My \rangle = \lambda' \langle x, y \rangle$.
- Since $\lambda \neq \lambda'$, we have $\langle x, y \rangle = 0$. 
Theorem (Courant-Fischer Theorem)

Let $M \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Let $\lambda_1 \leq \cdots \leq \lambda_n$ be a sequence of non-decreasing eigenvalues with multiplicities. Let $x_1, \ldots, x_i$ be the eigenvectors corresponding to $\lambda_1, \ldots, \lambda_i$. Then

$$\lambda_{k+1} = \min_{x \in \mathbb{R}^n \setminus \{0\} : x \perp \langle x_1, \ldots, x_k \rangle} \frac{\langle x, Mx \rangle}{\langle x, x \rangle}$$
Proof

- Using Spectral Theorem: The eigenvectors form an orthonormal basis
- If \( x \perp \langle x_1, \ldots, x_k \rangle \) then \( x = \sum_{n \geq i > k} a_i x_i \)
- Then we have:

\[
\frac{\langle x, Mx \rangle}{\langle x, x \rangle} = \frac{\sum_{i > k} a_i^2 \lambda_i}{\sum_{i > k} a_i^2} \geq \lambda_{k+1}
\]

Corollary

If \( \lambda_1 \leq \cdots \leq \lambda_n \) then

\[
\lambda_k = \min_{\text{dim}(V) = k} \max_{x \in V \setminus \{0\}} \frac{\langle x, Mx \rangle}{\langle x, x \rangle}
\]
Definition (Laplacian)

Let $G$ be a $d$-regular undirected graph with adjacency matrix $A$. The normalized Laplacian is defined to be:

$$L := I - \frac{1}{d} \cdot A$$

Note that $\langle x, Lx \rangle = \frac{1}{d} \sum_{(u,v) \in E} (x_u - x_v)^2$
Suppose the eigenvalues of $L$ are $\lambda_1 \leq \cdots \leq \lambda_n$. Then:

- $\lambda_1 = 0$ and $\lambda_n \leq 2$
- $\lambda_k = 0$ if and only if $G$ has $\geq k$ connected components
- $\lambda_n = 2$ if and only if a connected component of $G$ is bipartite
Proof

- Note that $\langle x, Lx \rangle \geq 0$, for all $x \in \mathbb{R}^n - 0$
- Note that a constant vector is a eigenvector with eigenvalue 0
- Therefore, $\lambda_1 = 0$

- If $\lambda_k = 0$ then there exists a vector space $V$ such that for any $x \in V$ we have $\sum_{(u,v) \in E} (x_u - x_v)^2 = 0$
- So, $x$ is constant within each component
- $k = \dim(V) \leq$ number of connected components in $G$
- Converse if easy to see using constant functions over each connected component
Note that $2\langle x, x \rangle - \langle x, Lx \rangle = \frac{1}{d} \sum_{(u,v) \in E} (x_u + x_v)^2$

So, $\lambda_n \leq 2$

Suppose $\lambda_n = 2$

Consider $x$ as its corresponding eigenvector

There exists an edge $(u, v)$ such that $x_u = a$ and $x_v = -a$

Let $A$ be the set $\{v : x_v = a\}$

Let $B$ be the set $\{v : x_v = -a\}$

Note that no edge connects two vertices within $A$ or two vertices within $B$

Note that no edge connects any vertex in $A$ with a vertex outside $B$

Note that no edge connects any vertex in $B$ with a vertex outside $A$

$(A, B)$ form a connected component and is bi-partite