Lecture 09: Spectral Graph Theory

Sparsest Cuts

Let $G = (V, E)$ be an undirected graph

Definition (Sparsity of a Cut)

$$\sigma(S) := \frac{\mathbb{E}_{(u,v)\in E}\left[|1_S(u) - 1_S(v)|\right]}{\mathbb{E}_{(u,v)\in V^2}\left[|1_S(u) - 1_S(v)|\right]}$$

Definition (Sparsity of a Graph)

$$\sigma(G) := \min_{S \subseteq V:\ S\neq\emptyset, S\neq V} \sigma(S)$$

For a d-regular graph G, $\sigma(S) = \frac{E(S,V-S)}{d|S||V-S|/|V|}$

Edge Expansion

Let G be a d-regular undirected graph

Definition (Edge Expansion of a Set)

$$\phi(S) := \frac{E(S, V - S)}{d\,|S|}$$

Definition (Edge Expansion of a Graph)

$$\phi(G) := \min_{S\colon |S| \leqslant |V|/2} \phi(S)$$

Relation between Sparsity and Edge Expansion

Lemma

For a regular graph G,

$$\frac{1}{2}\sigma(G) \leqslant \phi(G) \leqslant \sigma(G)$$

Proof is trivial

Expander Graph Family

Definition (Family of Expander Graphs)

Let $\{G_n\}_{n>d}$ be a family of d-regular graphs such that $\phi(G_n) \geqslant \phi$. This family is called a family of expander graphs.

Connectivity of Expanders

Lemma

Let $\phi(G) \geqslant \phi > 0$. Consider any $0 < \varepsilon < \phi$. On removal of any $\varepsilon |E|$ edges from G, there exists a connected component of G of size at least $(1 - \varepsilon/2\phi) |V|$.

- Let E' be any set of at most $\varepsilon |E|$ edges in G
- Let $C_1, \ldots, C_t$ be the connected components of G' (the graph obtained from G by removal of edges E') in non-decreasing order
- If $|C_1| \leqslant |V|/2$:

$$\begin{aligned} |E'| &\geqslant \frac{1}{2}\sum_{i \neq j} E(C_i, C_j) = \frac{1}{2}\sum_{i} E(C_i, V - C_i) \\ &\geqslant \frac{1}{2}\sum_{i} |C_i| \phi = \frac{d |V| \phi}{2} \end{aligned}$$

This is impossible

Proof Continued...

- If $|C_1| > |V|/2$:

$$\left|E'\right| \geqslant \mathbb{E}(C_1, V - C_1) \geqslant d\phi\,|V - C_1|$$
$$\implies |V - C_1| \leqslant \frac{\varepsilon d\,|V|}{2d\phi}$$

Hence, $|C_1| \geqslant (1 - \varepsilon/2\phi)\,|V|$

Hermitian Matrices

- $\langle x, y \rangle := x^* y = \sum_i \overline{x_i} y_i$
- $\langle x, x \rangle = \|x\| 2$
- A Hermitian matrix $M \in \mathbb{C}^{n \times n}$ satisfies $M = M^*$
- If $Mx = \lambda x$ then λ is the eigenvalue and x is the corresponding eigenvector

Eigenvalues of Hermitian Matrices

Lemma

All eigenvalues of a Hermitian M are real

- Suppose λ is an eigenvalue and x is its corresponding eigenvector
- Consider $\langle Mx, x\rangle = \langle x, M^*x\rangle = \langle x, Mx\rangle$
- Note that $\langle Mx, x\rangle = \overline{\lambda}\langle x, x\rangle$ and $\langle x, Mx\rangle = \lambda\langle x, x\rangle$
- Hence, $\overline{\lambda} = \lambda$

Eigenvectors of Hermitian Matrices are Orthogonal

Lemma

Let x and y be eigenvectors of a Hermitian M corresponding to two different eigenvalues. Then, $\langle x, y \rangle = 0$.

- Let λ and λ' be eigenvalues corresponding to x and y respectively
- Note that $\langle Mx, y \rangle = \lambda \langle x, y \rangle$
- Note that $\langle x, My \rangle = \lambda' \langle x, y \rangle$
- Since $\lambda \neq \lambda'$, we have $\langle x, y \rangle = 0$

Variational Characterization of Eigenvalues

Theorem (Courant-Fischer Theorem)

Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Let $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ be a sequence of non-decreasing eigenvalues with multiplicities. Let $x_1, \dots, x_i$ be the eigenvectors corresponding to $\lambda_1, \dots, \lambda_i$. Then

$$\lambda_{k+1} = \min_{x \in \mathbb{R}^n - \{0\}\colon x \perp \langle x_1, \dots, x_k \rangle} \frac{\langle x, Mx \rangle}{\langle x, x \rangle}$$

Proof

- Using Spectral Theorem: The eigenvectors form an orthonormal basis
- If $x \perp \langle x_1, \ldots, x_k \rangle$ then $x = \sum_{n \geqslant i > k} a_i x_i$
- Then we have:

$$\frac{\langle x, Mx \rangle}{\langle x, x \rangle} = \frac{\sum_{i>k} a_i^2 \lambda_i}{\sum_{i>k} a_i^2} \geqslant \lambda_{k+1}$$

Corollary

If $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ *then*

$$\lambda_k = \min_{\dim(V)=k} \quad \max_{x \in V - \{0\}} \frac{\langle x, Mx \rangle}{\langle x, x \rangle}$$

Definition (Laplacian)

Let G be a d-regular undirected graph with adjacency matrix A. The normalized Laplacian is defined to be:

$$L := I - \frac{1}{d} \cdot A$$

Note that $\langle x, Lx \rangle = \frac{1}{d} \sum_{(u,v) \in E} (x_u - x_v)^2$

Graph Properties from Laplacian

Theorem

Suppose the eigenvalues of L are $\lambda_1 \leqslant \cdots \leqslant \lambda_n$. *Then:*

- $\lambda_1 = 0$ *and* $\lambda_n \leqslant 2$
- $\lambda_k = 0$ *if and only if G has* $\geqslant k$ *connected components*
- $\lambda_n = 2$ *if and only if a connected component of G is bipartite*

Proof

- Note that $\langle x, Lx \rangle \geqslant 0$, for all $x \in \mathbb{R}^n - 0$
- Note that a constant vector is a eigenvector with eigenvalue 0
- Therefore, $\lambda_1 = 0$

- If $\lambda_k = 0$ then there exists a vector space V such that for any $x \in V$ we have $\sum_{(u,v)\in E}(x_u - x_v)^2 = 0$
- So, x is constant within each component
- $k = \dim(V) \leqslant$ number of connected components in G
- Converse if easy to see using constant functions over each connected component

Proof Continued

- Note that $2\langle x, x\rangle - \langle x, Lx\rangle = \frac{1}{d}\sum_{(u,v)\in E}(x_u + x_v)^2$
- So, $\lambda_n \leqslant 2$

- Suppose $\lambda_n = 2$
- Consider x as its corresponding eigenvector
- There exists an edge (u, v) such that $x_u = a$ and $x_v = -a$
- Let A be the set $\{v\colon x_v = a\}$
- Let B be the set $\{v\colon x_v = -a\}$
- Note that no edge connects two vertices within A or two vertices within B
- Note that no edge connects any vertex in A with a vertex outside B
- Note that no edge connects any vertex in B with a vertex outside A
- (A, B) form a connected component and is bi-partite