Lecture 07: Lovász Local Lemma and Moser-Tardos Algorithm

Lovász Local Lemma

- Let $\{X_1, \ldots, X_m\}$ be independent random variables
- Let $\{A_1, \ldots, A_n\}$ be events that are completely determined by evaluations of $(X_1, \ldots, X_n)$
- Intuition: A_is are bad events and we are interested in avoiding all of them

Theorem (Lovász Local Lemma)

Let $\Pr[A_i] \leqslant p < 1$ and each event A_i is independent of all but at most d of the other A_j events. If $ep(d+1) \leqslant 1$, then

$$\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0$$

The condition $ep(d+1) \leqslant 1$ can also be replaced with $4pd \leqslant 1$.

Let φ be a k-SAT formula with m variables and each variable occurs in at most $2^{k-2}/k$ clauses. Then φ is satisfiable.

- Consider a uniform random assignment of $X_j \xleftarrow{\$} \{\mathrm{T}, \mathrm{F}\}$
- Let there be n clauses and A_i represent the event that clause i is *not* satisfied
- We have $\Pr[A_i] = 2^{-k} =: p$
- And, $d \leqslant k \cdot 2^{k-2}/k$
- Then, $4pd \leqslant 1$
- Therefore, $\Pr[\wedge_i \overline{A_i}] > 0$ and, consequently, there exists a satisfying assignment

Application: Graph Coloring

Let G be a graph with maximum degree Δ. There exists a valid coloring of G with $C \geqslant 8(\Delta - 1)$ colors.

- For every edge e define A_e as the event that both vertices receive identical color
- Consider a uniformly random coloring
- $\Pr[A_e] = 1/C := p$
- $d \leqslant 2(\Delta - 1)$
- We have $4pd \leqslant 1$ when $C \geqslant 8(\Delta - 1)$

What is the bound when we use the following new sets of events: A_v is the event that some vertex in the neighborhood of v receives the same color as v.

- $\Pr[A_v] = \left(1 - \left(1 - \frac{1}{C}\right)^{\Delta}\right) =: p$
- $d = \Delta$

Think: When is $4pd \leqslant 1$?

Generalized Lovász Local Lemma

Let $\mathcal{G}$ be a graph where the events A_i are the vertices. We draw an edge (A_i, A_j) if A_i depends on A_j. This graph is called the *dependency graph*.

Theorem (Generalized Lovász Local Lemma)

If there exists a mapping $x\colon [m] \to (0,1)$ *such that: For all* $1 \leqslant i \leqslant n$*, we have*

$$\Pr[A_i] \leqslant x_i \prod_{j \in \Gamma_i} (1 - x_j)$$

Then $\Pr[\wedge_i \overline{A_i}] \geqslant \prod_i (1 - x_i) > 0$*.*

Proof

Trivially follows from the following technical lemma:

Lemma

For any $S \subseteq [n]$, we have: $\Pr[A_i | \wedge_{j \in S} \overline{A_j}] \leqslant x_i$

Because:

$$\begin{aligned} \Pr[\wedge_i \overline{A_i}] &= \prod_i \Pr[\overline{A_i} | \wedge_{j<i} \overline{A_j}] \\ &= \prod_i \left(1 - \Pr[A_i] | \wedge_{j<i} \overline{A_j}]\right) \\ &\geqslant \prod_i (1 - x_i) > 0 \end{aligned}$$

Proof of the Technical Lemma

- We proceed by induction on $|S|$
- Assume that for all S such that $|S| < t$ the hypothesis is true
- Consider an S such that $|S| = t$
- N_i is the set of all j such that A_i is not independent of A_j
- $S_1 = S \cap N_i$ and $S_2 = S \setminus S_1$
- Now, we have

$$\begin{aligned}\Pr[A_i | \wedge_{j \in S} \overline{A_j}] &= \Pr[A_i | \wedge_{j \in S_1} \overline{A_j}, \wedge_{j \in S_2} \overline{A_j}] \\ &= \frac{\Pr[A_i, \wedge_{j \in S_1} | \wedge_{j \in S_2} \overline{A_j}]}{\Pr[\wedge_{j \in S_1} \overline{A_j} | \wedge_{j \in S_2} \overline{A_j}]}\end{aligned}$$

- Let E_1 be the numerator and E_2 be the denominator

Proof of the Technical Lemma

- We have:

$$E_1 \leqslant \Pr[A_i | \wedge_{j \in S_2} \overline{A_j}] = \Pr[A_i]$$
$$\leqslant x_i \prod_{j \in N_i} (1 - x_j) \leqslant x_i \prod_{j \in S_1} (1 - x_j)$$

- Let $S_1 = \{j_1, \dots, j_k\}$
- And (by inductive hypothesis):

$$E_2 = \prod_{\ell \in [k]} \Pr[\overline{A_{j_\ell}} | \wedge_{\ell' < \ell} \overline{A_{j_{\ell'}}}, \wedge_{j \in S_2} \overline{A_j}]$$
$$= \prod_{\ell \in [k]} \left(1 - \Pr[\overline{A_{j_\ell}} | \wedge_{\ell' < \ell} \overline{A_{j_{\ell'}}}, \wedge_{j \in S_2} \overline{A_j}]\right)$$
$$\geqslant \prod_{\ell \in [k]} (1 - x_{j_\ell}) = \prod_{j \in S_1} (1 - x_j)$$

- Therefore, $E_1 / E_2 \leqslant x_i$

$\mathsf{vbl}(A_i)$ represents the variable on which even A_i depends on

function Seq_LLL($\mathcal{X} = \{X_1, \dots, X_m\}, \mathcal{A} = \{A_1, \dots, A_n\}$)
  $\mathcal{X} \leftarrow$ Random Evaluation
  while $\exists i$ s.t. A_i is satisfied **do**
    Pick arbitrary A_i that is satisfied
    $\mathcal{X}_{\mathsf{vbl}(A_i)} \leftarrow$ Random Evaluation
  end while
  Output $\mathcal{X}$
end function

Performance of Moser-Tardos

Theorem

Suppose there exists a mapping $x \colon \mathcal{A} \to (0,1)$ *such that, for all* $i \in [n]$*, we have* $\Pr[A_i] \leqslant x_i \prod_{j \in N_i} (1 - x_j)$*. Then the expected number of times sequential Moser-Tardos samples the event* A_i *is at most* $x_i/(1 - x_i)$ *and, hence, the expected number of execution of the inner loop is at most* $\sum_{i \in [n]} x_i/(1 - x_i)$*.*

function Parallel_LLL($\mathcal{X}, \mathcal{A}$)
    $\mathcal{X} \leftarrow$ Random Evaluation
    while $\exists i$ s.t. A_i is satisfied **do**
        Let S be a maximal independent set in the dependency graph restricted to satisfied A_is
        $\mathcal{X}_{\mathrm{vbl}(A_S)} \leftarrow$ Random Evaluation
    end while
    Output $\mathcal{X}$
end function

Performance of Parallel Moder-Tardos

Theorem

Suppose there exists an $\varepsilon > 0$ *and a mapping* $x\colon \mathcal{A} \to (0,1)$ *such that* $\Pr[A_i] \leqslant (1-\varepsilon)x_i \prod_{j \in N_i}(1-x_j)$. *The expected number of inner loops before all events in* $\mathcal{A}$ *are avoided is at most* $O\left(\frac{1}{\varepsilon}\sum_{i\in[n]} \frac{x_i}{1-x_i}\right)$.