Lecture 01: Pigeonhole Principle

Theorem (PHP)

For any placement of $(kn + 1)$ pigeons in n holes, there exists a hole with at least $(k + 1)$ pigeons.

Monochromatic Triangles in 2-Colorings

Theorem

Any 2-coloring of K_6 contains a monochromatic triangle.

- If possible let there exists a 2-coloring of K_6 that contains no monochromatic triangles
- Consider any vertex v in K_6
- There are 5 edges in K_6 that are incident on v
- By PHP, at least 3 of them have the same color
- Let edges (v, a), (v, b) and (v, c) are colored red
- Now, (a, b) must be colored blue (otherwise $\{v, a, b\}$ forms a monochromatic triangle)
- Similarly, (b, c) and (c, a) must be colored blue
- Then, $\{a, b, c\}$ forms a monochromatic triangle
- Hence, contradiction

Tightness

- Think: Give a 2-coloring of K_5 that has no monochromatic triangles

Theorem

Any 2-coloring of K_6 contains 2 monochromatic triangles.

- Define: A *biangle centered at b* is a set $\{a, b, c\}$ such that the edge (a, b) and (b, c) has different colors
- If possible, consider a coloring of K_6 with at most 1 monochromatic triangle
- There are $\binom{6}{3} = 20$ triangles in K_6
- A monochromatic triangle has 0 biangles
- A non-monochromatic triangle has 2 biangles
- This coloring has at least $20 - 1 = 19$ non-monochromatic triangles and, hence, at least 38 biangles

Tightness

Proof Continued...

- By PHP, there exists a vertex v such that it has at least 7 biangles centered at v
- But in K_6, any vertex either has 0, 4 or 6 biangles centered at it
- Hence, contradiction

- Think: Construct a 2-coloring for K_6 that has exactly 2 monochromatic triangles
- Think: Prove that any 2-coloring of K_7 has at least 4 monochromatic triangles

- Previous results are stepping stones to Ramsey Theory
- A Mathematical Gem:

Theorem (Van der Waerden Theorem)

For any r, k, there exists n such that any r-coloring of $\{1, \ldots, n\}$ has a monochromatic arithmetic progression of length k.

Erdös–Szekeres theorem

Theorem (Erdös–Szekeres Theorem)

Any set of distinct numbers $\{a_1, \ldots, a_n\}$ contains either an increasing subsequence of length $(a+1)$ or a decreasing subsequence of length $(b+1)$, where $n = ab+1$.

- Define the mapping $a_i \mapsto (u_i, v_i)$, where
  - u_i is the length of the longest increasing subsequence in $\{a_1, \ldots, a_i\}$ that includes a_i, and
  - v_i is the length of the longest decreasing subsequence in $\{a_1, \ldots, a_i\}$ that includes a_i.
- Suppose $\{a_1, \ldots, a_n\}$ has increasing subsequences of length at most a and decreasing subsequences of length at most b
- So, for all $i \in [n]$, we have $1 \leqslant u_i \leqslant a$ and $1 \leqslant v_i \leqslant b$
- There are at most ab distinct possible tuples (u_i, v_i)
- By PHP, there exists $i < j$ such that $(u_i, v_i) = (u_j, v_j)$

Erdös–Szekeres theorem

Proof Continued...

- If $a_j > a_i$ then $u_j > u_i$ (consider the longest increasing subsequence in $\{a_1, \ldots, a_i\}$ that ends in a_i and append a_j to it)
- If $a_j < a_i$ then $v_j > v_i$ (similarly)
- Therefore, it is not possible for $(u_i, v_i) = (u_j, v_j)$, for $i < j$
- Hence, contradiction

- Think: (Tightness) Construct a set of ab elements that has increasing subsequences of length at most a and decreasing subsequences of length at most b

Application

Let S_n be the set of all permutations of the set $[n]$. The expression $\pi \xleftarrow{\$} S_n$ represents a permutation drawn uniformly at random from S_n. Let $\text{inc}(\pi)$ denote the length of the longest increasing subsequence in the permutation π.

Theorem

$$\mathop{\mathbb{E}}_{\pi \xleftarrow{\$} S_n} [\text{inc}(\pi)] \geqslant \frac{\sqrt{n-1}}{2} + 1$$

- Note that π either has an increasing or decreasing subsequence of length $\sqrt{n-1}+1$
- So, π or reverse of π has an increasing sequence of length at least $\sqrt{n-1}+1$
- The other of the two permutations has an increasing sequence of length at least 1
- So, the expected length of the longest increasing sequence over π and reverse of π is $\frac{\sqrt{n-1}}{2}+1$

Food for Thought

1. Think: Prove $\mathbb{E}_{\pi \xleftarrow{\$} S_n}[\mathrm{inc}(\pi)] = \Theta\left(\sqrt{n}\right)$
2. Think: How does the distribution $\mathrm{inc}(\pi)$ look, for $\pi \xleftarrow{\$} S_n$?
3. Think: How to show that the distribution is strongly concentrated around its mean with variance $\approx n^{1/4}$?

PHP as Probability

Let M be a matrix. Let $M(r,c) \in [0,\infty)$ be the entry corresponding to the row r and column c. Let R and C be some distribution over the rows and columns respectively. The expression $r \sim R$ represents that the row r is drawn according to the distribution R and the expression $c \sim C$ represents that the column c is drawn according to the distribution C.

Theorem

Suppose

$$\mathop{\mathbb{E}}_{\substack{r\sim R\\ c\sim C}} [M(r,c)] \leqslant \varepsilon$$

If $\varepsilon = \alpha\beta$ *then,*

$$\Pr_{c\sim C}\left[\mathop{\mathbb{E}}_{r\sim R} [M(r,c)] \geqslant \alpha \right] \leqslant \beta$$

- Think: Prove it
- Think: How our first PHP is a special case of this?