

AKS PRIMALITY TESTING THE ALGORITHM, THEN THE PROOF

ABSTRACT. AKS tests whether an integer is prime in time polynomial in the number of input bits, with no random choices. The starting point is a polynomial identity that every prime satisfies. The algorithm keeps these polynomials short enough to compute with, then checks a small list of identities. Why is that enough? If a composite input passed every check, the proof would produce a finite collection of values that is both too large and too small. We explain the whole strategy first, state the algorithm, and then build this counting argument one step at a time. The supporting algebra and the bound on the running time are proved along the way.

1. INTRODUCTION: WHAT ARE WE TRYING TO ACCOMPLISH?

Give an algorithm the integer $N = 31$. It should answer “prime.” Give it $N = 35$, and it should answer “composite.” Trial division already does this. The interesting question is how the running time grows when N has hundreds or thousands of bits.

Throughout, N is the integer being tested and $n = \lfloor \log_2 N \rfloor + 1$ is its bit length. After the preliminary checks handle powers of two, we have

$$2^{n-1} < N < 2^n.$$

All logarithms below are base 2. A bound polynomial in n is the right notion of efficiency. A bound such as $O(\sqrt{N})$ is exponential in the input length.

Agrawal, Kayal, and Saxena proved that primality has a deterministic polynomial-time algorithm [AKS04]. Here is the version we will analyze. Its parameters are rounded a little more generously than those in the published algorithm; this makes the proof easier to read.

Main result

Theorem 1 (AKS, with rounded parameters). *Algorithm 1 returns PRIME exactly when its input $N \geq 2$ is prime. It is deterministic and uses $O(n^{17})$ bit operations with ordinary schoolbook arithmetic.*

This is a deliberately loose running-time bound. The important point is that the time is polynomial in n , the number of input bits. No random choices are needed; this is what *deterministic* means here.

The entire strategy, before the details. Let’s separate three questions: what should we test, how can we afford the test, and why can’t a composite number fool it?

- 1. Start with an exact test that is too big to run.** For a prime N , expanding $(X + a)^N$ modulo N leaves just $X^N + a$. In fact, with $a = 1$, this identity already distinguishes primes from composites. The catch is its size: a degree- N polynomial can need $N + 1$ coefficients, far too many for an n -bit input.
- 2. Keep only a short array of coefficients.** Choose a small integer r and use the rule $X^r = 1$. Exponents wrap around modulo r , so every calculation fits in an array of r coefficients. We lose information this way, so one shortened test is no longer enough. AKS checks several choices of a .
- 3. Choose r so that enough information survives.** Choose r sharing no factor greater than 1 with N , and require $1, N, N^2, \dots, N^{n^2}$ to stay distinct modulo r . This gives the proof many distinct points at which to compare polynomials. We will show that a suitable $r \leq 2n^5$ always exists, so finding r and storing the arrays take only polynomial resources.

4. **Handle simple ways to be composite first.** Check whether $N = u^v$ with $u, v \geq 2$, and compute greatest common divisors (gcds) to look for small factors. Once these checks pass, a composite N must have at least two different prime divisors, all larger than r .
5. **Show that such a composite cannot pass every test.** Suppose it did. Choose a prime divisor p just for the proof; the algorithm never needs to find it. In arithmetic based on this prime, the successful tests let us count one collection of products in two ways. First, choosing which factors to include gives many different values: an accidental equality would force two low-degree polynomials to agree at too many points. Second, the two different prime divisors force all those values to satisfy one equation with too few solutions. The two counts contradict each other.
6. **Put correctness and running time together.** Primes pass every check. The preceding argument says that a number passing the polynomial tests and the small-factor checks must be a power of a prime. The earlier power check leaves only a prime itself. Finally, repeated squaring runs each polynomial test efficiently, and both the array length and the number of tests are polynomial in n .

The heart of the proof is Step 5. Here is the contradiction in symbols; we will introduce the collection and the parameter k when we need them.

The counting argument we will prove

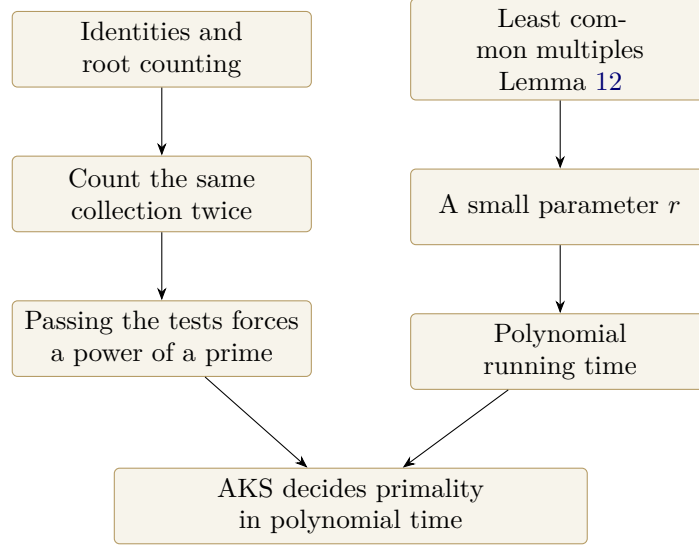
Suppose a composite input survives the preliminary checks and passes all the polynomial tests. Using a suitable prime divisor of the input, we construct a finite collection G of nonzero values. For an integer $k \geq 1$, the two counts give

$$|G| \geq N^k \quad \text{and} \quad |G| < N^k.$$

The first inequality comes from choosing which factors to include in a product. The second comes from two exponents having the same remainder. Both use one basic fact: over a field, a nonzero degree- d polynomial has at most d roots. Here a root is an input where the polynomial is zero; a field is an arithmetic system where division by any nonzero element is allowed.

How to read the note. Section 2 gives the complete algorithm and its data representation. Section 3 proves the main theorem using two precise intermediate results. We then open those results one at a time: the counting argument, the polynomial identities behind it, the small parameter r , and the running time. The extra arithmetic used in the proof is built explicitly in Section 7. It is never computed by the algorithm.

We use basic modular arithmetic, the Euclidean algorithm, the binomial theorem, and prime factorization as mathematical facts. We do not assume that the algorithm can factor its input. No prior course in finite fields is needed: the field facts used in the proof are established here.



An arrow $A \rightarrow B$ means that A is used to prove B .

2. THE ALGORITHM AND THE OBJECTS IT COMPUTES

2.1. A polynomial version of Fermat's theorem. For a prime p ,

$$(2.1) \quad (X + a)^p = X^p + a \pmod{p}$$

for every integer a . Equality here means equality of every coefficient modulo p . For example,

$$(X + 1)^5 = X^5 + 5X^4 + 10X^3 + 10X^2 + 5X + 1 = X^5 + 1 \pmod{5}.$$

For $N = 6$, the middle coefficients do not disappear:

$$(X + 1)^6 = X^6 + 3X^4 + 2X^3 + 3X^2 + 1 \pmod{6}.$$

In fact, $(X + 1)^N = X^N + 1 \pmod{N}$ holds exactly when N is prime; we prove this in Lemma 7. But its degree is N , so explicitly storing it would require exponentially many coefficients. AKS starts with this identity and keeps the polynomial calculations short [AKS04, Lemma 2.1 and equation (2), p. 783].

2.2. Keep polynomials short by wrapping exponents around. Fix a positive integer r . Working modulo $X^r - 1$ means using the rule $X^r = 1$. Every polynomial reduces to exactly one polynomial of degree less than r . For instance, if $r = 3$,

$$X^8 + 4X^4 + 2 = X^2 + 4X + 2 \pmod{X^3 - 1}.$$

For a compact name for this arithmetic, we write

$$R_{N,r} = (\mathbb{Z}/N\mathbb{Z})[X]/(X^r - 1).$$

This notation records two rules: reduce coefficients modulo N , and use $X^r = 1$. An element is just an array of r coefficients modulo N . To multiply two arrays, multiply polynomials and send the coefficient of X^j to position $j \bmod r$. Nothing needs to be divided by a coefficient. In particular, this arithmetic works whether or not N is prime.

The test we can afford is

$$(2.2) \quad (X + a)^N = X^N + a \quad \text{in } R_{N,r}.$$

Repeated squaring computes the left side using $O(n)$ multiplications of arrays of length r . The right side has a 1 in position $N \bmod r$ and an a in position 0, with addition if these positions coincide.

2.3. Choose r so that powers do not repeat too soon. Suppose $\gcd(N, r) = 1$. Start at 1 and repeatedly multiply by N , reducing modulo r each time. How many steps does it take to return to 1? This number is called the *multiplicative order*, written $\text{ord}_r(N)$: it is the smallest positive j with $N^j = 1 \pmod{r}$. It exists because multiplication by N rearranges the finitely many residues modulo r and can be undone using its inverse. AKS asks for

$$(2.3) \quad \text{ord}_r(N) > n^2.$$

This says that $1, N, N^2, \dots, N^{n^2}$ are distinct modulo r . Those distinct exponents will become distinct roots in the correctness proof.

To check (2.3), there is no need to factor r or compute its exact order. Compute $N, N^2, \dots, N^{n^2} \pmod{r}$ and check that none is 1.

Algorithm 1: AKS (rounded parameters)

Input: an integer $N \geq 2$. Set $n = \lfloor \log N \rfloor + 1$.

1. If $N = 2$, return PRIME.
2. If $N = u^v$ for integers $u \geq 2, v \geq 2$, return COMPOSITE. Such a number is called a *perfect power*.
3. Find the least $r \geq 2$ such that $\gcd(N, r) = 1$ and $\text{ord}_r(N) > n^2$.
4. For every integer a with $2 \leq a \leq r$, compute $d = \gcd(a, N)$. If $1 < d < N$, return COMPOSITE.
5. If $N \leq r$, return PRIME.
6. Set $\ell = \lceil n\sqrt{r} \rceil$. For every $1 \leq a \leq \ell$, check

$$(X + a)^N = X^N + a \quad \text{in } R_{N,r}.$$
 If a check fails, return COMPOSITE.
7. Return PRIME.

The search in Step 3 terminates with $r \leq 2n^5$ for $N \geq 3$ (Theorem 3). The ceiling in Step 6 can be computed using integer arithmetic: ℓ is the least integer with $\ell^2 \geq n^2 r$.

For comparison, the published AKS algorithm uses $\log^2 N$ for the order threshold and $\lfloor \sqrt{\varphi(r)} \log N \rfloor$ tests [AKS04, p. 784]. Here $\varphi(r)$ counts the residues modulo r that have inverses. Our larger bounds remove this quantity from the algorithm and make a subset-counting proof possible. This is the same AKS construction with different numerical parameters.

3. PROVE THE MAIN THEOREM FIRST

Here are the two intermediate results that drive the proof. Their proofs come later, so we can first see how the whole argument fits together.

The correctness criterion

Theorem 2. Let $N \geq 3$, $n = \lfloor \log N \rfloor + 1$, and let $r \geq 2$ satisfy $\gcd(N, r) = 1$ and $\text{ord}_r(N) > n^2$. Suppose every prime divisor of N is greater than r . If all the identities (2.2) hold for $1 \leq a \leq \ell = \lceil n\sqrt{r} \rceil$, then N is a power of a prime.

The search for r stays small

Theorem 3. For every integer $N \geq 3$, some $r \leq 2n^5$ satisfies $\gcd(N, r) = 1$ and $\text{ord}_r(N) > n^2$.

Proof of Theorem 1, using these two results. A prime is always accepted. It is not a proper perfect power. Every gcd in Step 4 is either 1 or N , and the polynomial identities hold by (2.1). Thus no step rejects a prime.

An accepted input is prime. The case $N = 2$ is settled at Step 1. Suppose first that Step 5 accepts. If N were composite, it would have a prime divisor $p < N \leq r$. Step 4 would have found $1 < \gcd(p, N) = p < N$, a contradiction.

Otherwise, the algorithm reaches Step 6 with $N > r$. Since Step 4 found no proper divisor, every prime divisor of N exceeds r . Passing all the polynomial tests therefore makes N a prime power by Theorem 2. Step 2 has already ruled out every proper perfect power. The remaining possibility is that N itself is prime.

Finally, Theorem 3 bounds the search parameter by $2n^5$. The implementation analysis in Section 9 then gives the claimed $O(n^{17})$ bit bound. Every step uses a specified finite search or exact arithmetic, so the algorithm is deterministic. $\square$

What remains

The difficult part is Theorem 2. We will prove it by assuming that N has at least two distinct prime divisors and deriving the two incompatible bounds on $|G|$. After that, the search bound for r is an elementary argument about least common multiples.

4. THE CENTRAL ARGUMENT: ONE COLLECTION, TWO COUNTS

This section proves Theorem 2 using three lemmas whose statements appear before they are used. The two counting lemmas are proved in Section 5; the algebraic setup is proved in Section 6.

4.1. Choose a prime divisor for the proof. Assume the hypotheses of Theorem 2, and suppose for a contradiction that N is not a prime power. We may choose a prime divisor $p \mid N$ such that

$$(4.1) \quad p \not\equiv 1 \pmod{r}.$$

Indeed, if every prime divisor were $1 \pmod{r}$, their product with multiplicity would give $N \equiv 1 \pmod{r}$. That would make $\text{ord}_r(N) = 1$. Also, $p > r$ by hypothesis. Put $b = N/p$.

This is a choice inside the proof. The algorithm neither finds p nor uses it in a computation.

4.2. The values we will count. We want polynomial equations to behave predictably: a nonzero degree- d polynomial should have at most d roots. Arithmetic modulo a composite N does not guarantee this. Arithmetic modulo the prime p does. Write $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for it; this is a field because every nonzero residue has an inverse.

We will use a larger finite field K containing $\mathbb{F}_p$. The extra values let us choose an element ζ whose powers return to 1 for the first time after exactly r steps. In other words,

$$\zeta^r = 1, \quad \zeta^i = \zeta^j \iff i = j \pmod{r}.$$

Such a ζ is called a *primitive r th root of unity*. What matters to us is the displayed rule: different exponents modulo r give different powers of ζ . The next lemma guarantees this choice. Our collection will consist of products of $\zeta + 1, \dots, \zeta + \ell$.

We will occasionally call a collection a *multiplicative group*. This means it contains 1, products, and inverses, with the usual associative multiplication. The *order* of an element is the number of multiplications, starting from 1, before it first returns to 1.

Let $\mathcal{P}$ be the set of formal polynomials

$$f(X) = \prod_{a=1}^{\ell} (X + a)^{e_a}, \quad e_a \geq 0,$$

with coefficients in $\mathbb{F}_p$; the empty product is 1. Set

$$\mathcal{I} = \{b^i p^j : i, j \geq 0\}, \quad H = \{m \bmod r : m \in \mathcal{I}\}, \quad G = \{f(\zeta) : f \in \mathcal{P}\}.$$

Because b, p divide N and $\gcd(N, r) = 1$, both are invertible modulo r . Thus H is a group of invertible residues modulo r , and the next lemma shows that G is a multiplicative group in K . Why are nonnegative exponents enough? In a finite group, an element z eventually satisfies $z^u = 1$ for some $u \geq 1$, so its inverse is the nonnegative power z^{u-1} .

Lemma 4 (Algebraic setup). *There is a finite field K and a primitive r th root $\zeta \in K$ such that $\zeta \notin \mathbb{F}_p$. In particular, $\zeta + a \neq 0$ for every $a \in \mathbb{F}_p$, so G is a finite multiplicative group. For every $f \in \mathcal{P}$ and $m \in \mathcal{I}$,*

$$(4.2) \quad f(\zeta)^m = f(\zeta^m).$$

The identity says that two operations agree: first evaluate and then take the m th power, or first take the m th power of the input and then evaluate. This is how one equality will spread to many points.

Write $t = |H|$. Since $N = bp$, the group H contains all powers of N . Therefore

$$(4.3) \quad n^2 < \text{ord}_r(N) \leq t \leq r - 1.$$

Set

$$(4.4) \quad k = \lfloor \sqrt{t} \rfloor, \quad D = nk.$$

These parameters satisfy

$$(4.5) \quad D < t, \quad D \leq \ell, \quad D < p.$$

For the first inequality, $n < \sqrt{t}$ and $k \leq \sqrt{t}$, so $nk < t$. For the second, $nk \leq n\sqrt{r} \leq \ell$. The last follows from $D < t \leq r - 1 < p$.

Lemma 5 (Many products). *In this setup, $|G| \geq 2^D \geq N^k$.*

Lemma 6 (A small equation for every product). *If N is not a power of p , then $|G| < N^k$.*

Proof of Theorem 2. If N were not a prime power, choose p, b, K, ζ, H, G as above. In particular, N would not be a power of this chosen p . Lemma 5 gives $|G| \geq N^k$, while Lemma 6 gives $|G| < N^k$. The contradiction proves the theorem. $\square$

This is how the AKS proof fits together. The published version counts more general products and uses a binomial coefficient. Our rounded parameters let us count subsets instead; compare [AKS04, Lemmas 4.7–4.8, pp. 787–788] and [Sho08, Lemmas 21.9–21.10, pp. 556–558].

5. OPEN THE TWO COUNTING LEMMAS

We will use the following elementary fact several times: a nonzero polynomial of degree d over a field has at most d roots. A proof appears in Lemma 8. Notice that this statement applies over the field K ; it need not hold over $\mathbb{Z}/N\mathbb{Z}$ for composite N .

5.1. The lower bound: every subset gives something new.

Proof roadmap

Use the first D linear factors $X + 1, \dots, X + D$. Each subset gives one product of degree at most $D < t$. If two products have the same value at ζ , identity (4.2) makes them agree at t distinct points. Their difference would have more roots than its degree.

Proof of Lemma 5. For each subset $S \subseteq \{1, \dots, D\}$, define

$$f_S(X) = \prod_{a \in S} (X + a).$$

There are 2^D such products, and their degrees are at most $D < t$. They are distinct as formal polynomials over $\mathbb{F}_p$: if $S \neq T$, choose (after interchanging S, T if necessary) $a \in S \setminus T$. Then $f_S(-a) = 0$, whereas

$$f_T(-a) = \prod_{c \in T} (c - a) \neq 0.$$

Every factor is nonzero modulo p because $1 \leq a, c \leq D < p$ and $c \neq a$.

We claim that these products also have distinct values at ζ . Suppose $f_S(\zeta) = f_T(\zeta)$. For any $m \in \mathcal{I}$, raise this equality to the m th power and use (4.2) to obtain

$$f_S(\zeta^m) = f_T(\zeta^m).$$

As m ranges over $\mathcal{I}$, its residues range over the t elements of H . Since ζ has order r , these give t distinct points ζ^m . Thus $f_S - f_T$ is a nonzero polynomial of degree at most $D < t$ with at least t roots. This is impossible.

Every $f_S(\zeta)$ belongs to G , so

$$|G| \geq 2^D = 2^{nk} \geq N^k,$$

where the last inequality uses $2^n \geq N$. □

5.2. The upper bound: a collision of exponents.

Proof roadmap

Take a square grid of exponents $b^i p^j$, with $0 \leq i, j \leq k$. The grid contains more than t distinct integers, so two have the same residue in H . Their difference becomes an exponent that sends every element of G to 1. The root bound then makes G small.

Proof of Lemma 6. Because N is not a power of p , another prime $q \neq p$ divides $b = N/p$. The integers

$$b^i p^j, \quad 0 \leq i, j \leq k,$$

are all distinct. To check this, suppose $b^i p^j = b^{i'} p^{j'}$. Comparing the exponents of q gives $i = i'$. Cancelling b^i then gives $j = j'$.

There are $(k+1)^2 > t$ integers in this grid and only t possible residues in H . Hence there are $m_1 > m_2$ in the grid with $m_1 \equiv m_2 \pmod{r}$. Put $\delta = m_1 - m_2 > 0$.

Take any $z \in G$ and write $z = f(\zeta)$ for some $f \in \mathcal{P}$. Using (4.2) twice gives

$$z^{m_1} = f(\zeta^{m_1}) = f(\zeta^{m_2}) = z^{m_2}.$$

Since $z \neq 0$, division gives $z^\delta = 1$. Thus every element of G is a root of the nonzero polynomial $Y^\delta - 1$ over K , so $|G| \leq \delta$. Finally, every grid entry is at most

$$b^k p^k = (bp)^k = N^k.$$

Consequently

$$|G| \leq m_1 - m_2 < m_1 \leq N^k.$$

This is the required strict upper bound. □

Pause and check the mechanism. There are two different uses of the root bound. For the lower bound, it prevents two low-degree products from colliding at ζ . For the upper bound, it limits how many nonzero elements can satisfy $z^\delta = 1$. The same evaluation identity connects both uses.

The choice of $b = N/p$ is also doing real work. The largest grid entry is exactly $b^k p^k = N^k$, which matches the lower bound. Replacing this grid by $N^i p^j$ would give a weaker upper bound.

6. WHERE THE EVALUATION IDENTITY COMES FROM

We now prove Lemma 4. The field element ζ is constructed in Section 7; first we explain the identities that make it useful. The corresponding steps of the original proof appear in [AKS04, Definition 4.4 and Lemmas 4.5–4.6, pp. 786–787].

Proof roadmap

Reduce the successful tests modulo p . Then show that equal p th powers have equal bases in our array arithmetic, giving the identity for $b = N/p$. Extend it to products of factors and products of exponents, and finally evaluate at ζ . Each step preserves exactly the identity needed by the counting argument.

6.1. Taking a p th power modulo p . For $0 < j < p$, the binomial coefficient $\binom{p}{j}$ is divisible by p . The middle terms of the binomial expansion therefore vanish when we calculate modulo p :

$$(u + v)^p = u^p + v^p.$$

This works for numbers, polynomials, and our coefficient arrays: all use the same addition and multiplication rules, with coefficients modulo p . Also $a^p = a$ for $a \in \mathbb{F}_p$: apply this identity repeatedly to a sum of copies of 1. For a polynomial f with coefficients in $\mathbb{F}_p$, it follows that

$$(6.1) \quad f(X)^p = f(X^p).$$

Work now in

$$R = \mathbb{F}_p[X]/(X^r - 1).$$

Because $p > r$, we have $\gcd(p, r) = 1$. If $u = \sum_{j=0}^{r-1} c_j X^j$, then

$$u^p = \sum_{j=0}^{r-1} c_j X^{pj} \quad \text{in } R.$$

The map $j \mapsto pj \bmod r$ permutes the r coefficient positions. So taking a p th power only rearranges the array. In particular, if $u^p = v^p$, then $u = v$. This is why we are allowed to cancel a p th power below. We are not assuming that R is a field.

6.2. First obtain the identity for p and $b = N/p$. Reduce the algorithm's successful tests modulo the prime divisor p . For each tested factor $f(X) = X + a$, they give

$$f(X)^N = X^N + a \quad \text{in } R.$$

Since $N = pb$, we have

$$(f(X)^b)^p = X^{pb} + a = (X^b + a)^p.$$

The arrays inside these equal p th powers must be equal, giving

$$(6.2) \quad f(X)^b = f(X^b) \quad \text{in } R.$$

Equation (6.1) gives the same statement for the exponent p .

6.3. Extend the identity to all the products we need. We now know that

$$f(X)^m = f(X^m) \quad \text{in } R$$

for each tested factor and each exponent $m = b, p$. Two observations let us extend this equality.

- (1) If it holds for f and g with the same exponent m , multiplying the two identities gives it for fg .
- (2) If it holds for f with exponents m and m' , then it holds with exponent mm' . Indeed,

$$f(X)^{mm'} = f(X^m)^{m'} = f(X^{mm'}) \quad \text{in } R.$$

The last step substitutes X^m for X in the identity for m' .

This substitution is legitimate: a multiple of $X^r - 1$ becomes a multiple of $X^{mr} - 1$, which is still divisible by $X^r - 1$.

Starting with the tested factors and the exponents b, p , these observations give the equality for every $m \in \mathcal{I}$ and every $f \in \mathcal{P}$. Evaluating at any ζ with $\zeta^r = 1$ gives (4.2).

6.4. Finish the setup lemma. Section 7 constructs a finite field K containing $\mathbb{F}_p$ with a primitive r th root ζ . It cannot lie in $\mathbb{F}_p$. Otherwise $\zeta^p = \zeta$, so $\zeta^{p-1} = 1$, which would imply $r \mid p-1$, contrary to (4.1). Therefore $\zeta + a \neq 0$ for every $a \in \mathbb{F}_p$.

The set G contains 1, is closed under multiplication, and lies in the finite set of nonzero elements of K . It is itself a group: for $z \in G$, some positive power z^u equals 1, so z^{u-1} is an inverse in G . Together with the evaluation identity just proved, this establishes Lemma 4. $\square$

6.5. Why the original identity distinguishes primes from composites. The next lemma completes the motivation from Section 2. Its converse is not needed elsewhere in the correctness proof.

Lemma 7. *For an integer $N \geq 2$, the coefficient identity $(X + 1)^N = X^N + 1 \pmod{N}$ holds if and only if N is prime.*

Proof. The prime direction follows from the binomial calculation above. If N is composite, choose a prime $p \mid N$, and let p^e be the largest power of p dividing N . Then $p < N$, and

$$\binom{N}{p} = \frac{N}{p} \binom{N-1}{p-1}.$$

The second factor is $1 \pmod{p}$: its numerator is $(N-1) \cdots (N-p+1)$, whose factors reduce to $-1, \dots, -(p-1)$; division by $(p-1)!$ is valid modulo p . Thus exactly p^{e-1} divides $\binom{N}{p}$, and p^e does not. This middle coefficient is not divisible by N , so the identity fails. $\square$

7. BUILD THE FIELD USED IN THE PROOF

There is only one purpose for the field K : it provides r distinct powers of ζ and lets us use the polynomial root bound. Here are complete proofs of the facts we need. These correspond to the standard results in [Sho08, Theorems 7.14, 7.29, and 16.25].

A larger field containing F , with the same arithmetic on the elements already in F , is called a *field extension*. Think of including extra values so that equations missing solutions can acquire them.

Lemma 8 (Root bound). *A nonzero polynomial of degree d over a field has at most d roots.*

Proof. If a is a root, polynomial division gives $f(X) = (X - a)g(X)$ with $\deg g = d - 1$. Every other root is a root of g , since nonzero field elements can be cancelled. Induction on d proves the claim. $\square$

Lemma 9 (A finite field large enough to contain the roots). *Every nonzero polynomial over a finite field can be written as a constant times a product of linear factors after enlarging to another finite field.*

Proof. Keep factoring a nonconstant polynomial until one of its nonconstant factors cannot be written as a product of two positive-degree polynomials over the current field. Such a factor is called *irreducible*. This process stops because the degree decreases. If an irreducible factor $h(T)$ has degree $d > 1$, form $F[T]/(h)$: calculate with polynomials, keeping only their remainders on division by h . This is a field: for any nonzero representative g of degree less than d , irreducibility gives $\gcd(g, h) = 1$. Long division over a field gives the same Euclidean algorithm and Bézout identity as for integers. Thus $ug + vh = 1$ for some polynomials u, v , so u is an inverse of g modulo h .

This arithmetic has $|F|^d$ elements, represented by coefficient arrays of length d , and the element represented by T is a root of h .

Thus we can enlarge the field to include a root of any irreducible factor of degree greater than one. Divide out the resulting linear factor and repeat over the enlarged field. The remaining degree drops at each step, so after finitely many steps the whole polynomial is a constant times a product of linear factors. Each step keeps the field finite. $\square$

Lemma 10 (One element generates all the values). *Let U be a finite collection of nonzero field elements containing 1 and closed under multiplication and inverses. Some element of U has order $|U|$: its powers run through every value in U .*

Proof. Let e be the least common multiple of the orders of all elements of U . For each prime power q^a dividing e exactly, choose an element whose order is divisible by q^a but not by q^{a+1} . If that order is $q^a c$ with $\gcd(q, c) = 1$, raising the element to the c th power produces an element of order q^a .

Multiply these chosen elements, one for each prime divisor of e . The product has order e . Here is why. If field elements x, y have orders u, v with $\gcd(u, v) = 1$, their product has order uv : if $(xy)^s = 1$, then $x^s = y^{-s}$ is both a power of x and a power of y . Its order therefore divides both u and v , so this common value is 1; hence $u \mid s$ and $v \mid s$. Repeating proves the assertion for all the chosen factors.

Every element of U satisfies $z^e = 1$, so the root bound gives $|U| \leq e$. The element of order e has e distinct powers in U , giving $e \leq |U|$. Thus $e = |U|$, as required. $\square$

Lemma 11 (An element with exactly r distinct powers). *If p is prime and $\gcd(p, r) = 1$, some finite extension of $\mathbb{F}_p$ contains an element of order exactly r .*

Proof. By Lemma 9, choose a finite field in which $X^r - 1$ is a product of linear factors. Its roots are distinct. Indeed, a repeated root α would also be a root of its derivative rX^{r-1} , computed by the usual power rule: this follows by differentiating a factorization containing $(X - \alpha)^2$. But $\alpha \neq 0$ and $r \not\equiv 0 \pmod{p}$.

There are therefore exactly r roots. They contain 1, and their products and inverses are again r th roots. Lemma 10 supplies an element of order r . $\square$

This finishes the algebra used in the AKS argument. The field construction may be large, but it is an existence argument inside the proof. It contributes no operations to the algorithm.

8. WHY A SMALL PARAMETER r EXISTS

We turn to Theorem 3. The proof follows the corrected least-common-multiple approach to AKS: the original proof of its search lemma was corrected in 2019 [AKS19, pp. 317–318]. We include the whole argument, with constants adapted to our rounded n .

Proof roadmap

Suppose every candidate $r \leq B$ failed: whenever $\gcd(N, r) = 1$, one of $N, N^2, \dots, N^{n^2}$ would already be 1 modulo r . Then every prime power up to B would divide a single explicit integer Q . Consequently the least common multiple of $1, \dots, B$ would divide Q . But that least common multiple grows at least exponentially in B , while our formula for Q makes it too small. Set $B = 2n^5$ and compare the two bounds.

8.1. The elementary growth estimate.

Lemma 12. *For every integer $B \geq 1$,*

$$\text{lcm}(1, \dots, B) \geq 2^{B-1}.$$

Proof. Put $M = \text{lcm}(1, \dots, B)$. We first show that each of the integers

$$B \binom{B-1}{j} = \frac{B!}{j!(B-1-j)!}, \quad 0 \leq j \leq B-1,$$

divides M . For a prime q , the exponent of q in $s!$ is $\sum_{a \geq 1} \lfloor s/q^a \rfloor$: count multiples of q , then the extra factors from multiples of q^2 , and so on. Hence the exponent of q in the displayed integer is

$$\sum_{a \geq 1} \left(\left\lfloor \frac{B}{q^a} \right\rfloor - \left\lfloor \frac{j}{q^a} \right\rfloor - \left\lfloor \frac{B-1-j}{q^a} \right\rfloor \right).$$

Each summand is 0 or 1. To see this, write the two remainders of j and $B-1-j$ modulo q^a as u, v . The summand is $\lfloor (u+v+1)/q^a \rfloor$, and $0 \leq u+v+1 < 2q^a$. Also the summand is 0 when $q^a > B$. Thus the exponent is at most $\lfloor \log_q B \rfloor$, exactly the exponent of q in M . This proves the divisibility claim.

The B binomial coefficients in row $B-1$ sum to 2^{B-1} , so one is at least $2^{B-1}/B$. Since M is at least B times that coefficient, $M \geq 2^{B-1}$. $\square$

For another reference to this exact bound, see [Far09, Corollary 3, arXiv version 2, p. 4].

8.2. Apply the growth estimate.

Proof of Theorem 3. For $N \geq 3$ we have $n \geq 2$. Set

$$B = 2n^5, \quad e = \lfloor \log B \rfloor, \quad Q = N^e \prod_{i=1}^{n^2} (N^i - 1).$$

Suppose every $2 \leq r \leq B$ coprime to N had order at most n^2 . We claim that every prime power $q^a \leq B$ divides Q .

- If $q \mid N$, then $a \leq \lfloor \log B \rfloor = e$, so $q^a \mid N^e$.
- If $q \nmid N$, the assumed order bound for the modulus q^a gives some $1 \leq i \leq n^2$ with $q^a \mid N^i - 1$.

It follows that $\text{lcm}(1, \dots, B) \mid Q$.

On the other hand, since $\log N \leq n$ and $N^i - 1 < N^i$,

$$\begin{aligned} \log Q &< n \log B + n \sum_{i=1}^{n^2} i \\ &= n \log B + \frac{n^5 + n^3}{2} \\ &\leq n^5 + \frac{5}{8}n^5 = \frac{13}{8}n^5 < 2n^5 - 1 = B - 1. \end{aligned}$$

Here are the numerical checks. For integer $n \geq 2$, $\log n \leq n-1$, so $\log B = 1 + 5 \log n \leq 5n - 4 \leq 5n$. Therefore $n \log B \leq 5n^2 \leq n^5$. Also $n^3 \leq n^5/4$, and $(3/8)n^5 > 1$.

We have proved $Q < 2^{B-1}$, contradicting

$$2^{B-1} \leq \text{lcm}(1, \dots, B) \leq Q$$

from Lemma 12. Thus some required $r \leq 2n^5$ exists. $\square$

Notice why N^e was included in Q . It absorbs *all* prime powers up to B whose primes divide N . A single factor of N would not be enough. This is the issue addressed by the AKS erratum [AKS19].

9. RUNNING TIME AND AN IMPLEMENTATION VIEW

Proof roadmap

Count bits, not the numerical value of N . Every polynomial is stored as r residues with $O(n)$ bits each. The search bound gives $r = O(n^5)$, the number of tests is $O(n\sqrt{r})$, and repeated squaring uses $O(n)$ polynomial multiplications per test.

9.1. The preliminary work is polynomial. To test whether N is a perfect power, try each $2 \leq v \leq n$ and use binary search for an integer u with $u^v = N$. There is no need to try larger v , since $u \geq 2$ implies $2^v \leq N$. In a comparison, multiply with an early stop as soon as the value exceeds N . The integers kept have $O(n)$ bits. Even using $O(n)$ multiplications per comparison, $O(n)$ comparisons per search, and $O(n)$ exponents gives $O(n^5)$ bit operations with schoolbook multiplication.

For each $r \leq 2n^5$, compute $\gcd(N, r)$ and, if it is 1, check the first n^2 powers of $N \pmod{r}$. Each modulus has $O(\log n)$ bits. The loose bound $O(n^9)$ covers the whole search using ordinary integer arithmetic. The gcd checks through r cost at most $O(rn^3)$: Euclid's algorithm uses $O(n)$ divisions, each costing $O(n^2)$ by long division. For the division count, observe that every two remainders decrease by at least a factor of two. None of these steps requires factoring N or r .

9.2. Multiplying the coefficient arrays. For arrays $u = (u_0, \dots, u_{r-1})$ and $v = (v_0, \dots, v_{r-1})$, their product in $R_{N,r}$ has entries

$$w_k = \sum_{i+j \equiv k \pmod{r}} u_i v_j \pmod{N}.$$

The direct double loop uses r^2 coefficient multiplications. Reduce modulo N after each update, so all stored coefficients have $O(n)$ bits. One polynomial multiplication costs $O(r^2 n^2)$ bit operations. Repeated squaring therefore checks one identity in $O(r^2 n^3)$ bit operations.

There are

$$\ell = \lceil n\sqrt{r} \rceil = O(n^{7/2})$$

tests. Their total cost is

$$O(\ell r^2 n^3) = O(n^{33/2}) \subseteq O(n^{17}).$$

The preliminary work is smaller. This proves the running-time statement used in Theorem 1. Compare the schoolbook analysis in [Sho08, Theorem 21.3, p. 551]; we make no claim that our loose exponent is an optimized AKS bound.

What is computed, and what appears only in the proof

Algorithm	Correctness proof
n, r, ℓ ; gcds; modular powers; arrays of r coefficients	A prime divisor p ; $b = N/p$; a finite field K ; ζ ; the groups H, G

10. EXAMPLES AND A FEW USEFUL CHECKS

10.1. A full small trace: $N = 31$. Here $n = 5$, so the order threshold is 25. No $r \leq 26$ can work, because there are at most $r - 1 \leq 25$ nonzero residues through which its powers can run. For the next candidates,

$$\text{ord}_{27}(31) = 9, \quad \text{ord}_{28}(31) = 6, \quad \text{ord}_{29}(31) = 28.$$

Thus the search selects $r = 29$. For the last equality, $31 = 2 \pmod{29}$, $2^{14} = -1 \pmod{29}$, and $2^4 = 16 \not\equiv 1 \pmod{29}$; these rule out the proper divisors of 28 as the order.

The input is not a perfect power, and all gcds with $2, \dots, 29$ are 1. Since $31 > 29$, the algorithm reaches the polynomial tests, with $\ell = \lceil 5\sqrt{29} \rceil = 27$. For every tested a ,

$$(X + a)^{31} = X^{31} + a = X^2 + a \quad \text{in } R_{31,29}.$$

All tests pass, and the output is **PRIME**.

For contrast, $N = 15$ has $n = 4$ and the search finds $r = 19$. Step 4 detects $\gcd(3, 15) = 3$ and returns **COMPOSITE** before any polynomial test. The input $N = 49$ is rejected even earlier, as 7^2 .

10.2. A small polynomial calculation. For a concrete array example, work modulo 5 and $X^3 - 1$. Then

$$(X + 1)^2 = X^2 + 2X + 1, \quad (X + 1)^4 = X^2 \quad \text{in } R_{5,3}.$$

Multiplying once more by $X + 1$ gives

$$(X + 1)^5 = X^2 + 1 \quad \text{in } R_{5,3}.$$

For instance, the middle calculation starts from $X^4 + 4X^3 + 6X^2 + 4X + 1$ and wraps X^4 to X and X^3 to 1. This is an illustration of the arithmetic, not the parameter choice that AKS would use on input 5.

10.3. Why the preliminary checks matter.

- **Perfect powers.** The counting criterion concludes that N is a prime power. Step 2 converts that conclusion into primality. It also guarantees a second prime divisor when we need distinct integers in the exponent grid.
- **Small factors.** The gcd checks make $p > r$, so the short list $1, \dots, D$ stays distinct modulo p .
- **Large order.** It gives $t > n^2$, hence $D = n\lfloor\sqrt{t}\rfloor < t$. That strict degree inequality is what makes the lower bound work.
- **A suitable prime divisor.** Choosing $p \not\equiv 1 \pmod{r}$ ensures $\zeta \notin \mathbb{F}_p$, so none of the factors $\zeta + a$ is zero.

10.4. Exercises with short answers.

- (1) Why is testing all integers up to $\sqrt{N}$ not polynomial time in the input length? *Answer:* $\sqrt{N}$ is of size $2^{\Theta(n)}$.
- (2) Show that $\ell \leq r$ for our selected r . *Answer:* $r - 1 \geq \text{ord}_r(N) > n^2$, so $n\sqrt{r} < r$; taking a ceiling gives $\ell \leq r$.
- (3) Where is the fact that K is a field used in the upper bound? *Answer:* to cancel $z^{m_2} \neq 0$ and to apply the root bound.
- (4) Why is $(k + 1)^2 > t$ even when t is a square? *Answer:* $k = \lfloor\sqrt{t}\rfloor$, so $k + 1 > \sqrt{t}$.
- (5) Does this proof give a polynomial-time factoring algorithm? *Answer:* no. A failed polynomial identity certifies compositeness without necessarily revealing a factor.

11. TECHNICAL INPUTS AND PRECISE SOURCE LOCATIONS

All the AKS-specific inputs below have been proved in this note. These boxes record the exact statements used and their source locations, so the dependency chain can be checked without guessing which version of a result is intended. Page numbers for Shoup refer to the printed numbering in the author's Version 2 PDF linked in the bibliography.

Input 1: elementary arithmetic

Every integer greater than 1 has a unique factorization into primes. For integers u, v , the extended Euclidean algorithm produces coefficients expressing $\gcd(u, v)$ as an integer linear

combination of u, v . In particular, $\mathbb{Z}/p\mathbb{Z}$ is a field when p is prime. These are the elementary arithmetic prerequisites used throughout the note.

Precise references. Shoup, Theorem 1.3, p. 2 (proof completed on p. 8); Theorems 1.7–1.8, pp. 6–7; the extended Euclidean algorithm, Theorem 4.3, p. 78; and Example 7.11, p. 170 [Sho08].

Input 2: root counting and a field with primitive roots

A nonzero degree- d polynomial over a field has at most d roots. Every nonzero polynomial over a finite field splits in a finite extension. Every finite multiplicative subgroup of a field is cyclic. Consequently, if $p \nmid r$, a finite extension of $\mathbb{F}_p$ contains an element of order r .

Proofs here. Lemmas 8–11.

Precise references. Shoup: root bound, Theorem 7.14, p. 181; cyclicity, Theorem 7.29, p. 203; the irreducible quotient construction, §16.4, p. 437; splitting fields, Theorem 16.25, p. 443; the derivative criterion for repeated factors, Theorem 19.1, p. 509 [Sho08].

Input 3: the polynomial identities propagate

In $R = \mathbb{F}_p[X]/(X^r - 1)$ with $p \nmid r$, equal p th powers have equal bases: $f^p = g^p$ implies $f = g$. If a tested factor satisfies $f^N = f(X^N)$ and $p \mid N$, it also satisfies $f^{N/p} = f(X^{N/p})$. The equality $f^m = f(X^m)$ continues to hold when we multiply polynomials or multiply exponents for which it already holds.

Proof here. Section 6, using the explicit permutation $j \mapsto pj \bmod r$ of coefficient positions.

Precise references. AKS, Definition 4.4 and Lemmas 4.5–4.6, pp. 786–787 [AKS04]; Shoup, Lemmas 21.7–21.8, p. 555, and Frobenius in Example 7.48, p. 194, and Theorem 19.7, p. 512 [Sho08]. The implication $f^p = g^p \Rightarrow f = g$ in our array arithmetic follows from the coefficient-rearrangement proof above.

Input 4: the two counting estimates

With the parameters of Section 4, $k = \lfloor \sqrt{t} \rfloor$ and $D = nk$ give $|G| \geq 2^D \geq N^k$. If N is not a power of p , two distinct grid entries $b^i p^j$, $0 \leq i, j \leq k$, collide modulo r , and yield $|G| < N^k$.

Proofs here. Lemmas 5–6.

Precise references. AKS, Lemmas 4.7–4.8, pp. 787–788, and the concluding Lemma 4.9, pp. 788–789 [AKS04]. Shoup gives a related subset argument in Lemmas 21.9–21.11, pp. 556–558 [Sho08]. The numerical inequalities stated here use our rounded parameters; they are proved directly rather than quoted from either source.

Input 5: a small search parameter

For $B \geq 1$, $\text{lcm}(1, \dots, B) \geq 2^{B-1}$. For $N \geq 3$ and $n = \lfloor \log_2 N \rfloor + 1$, there is an $r \leq 2n^5$ coprime to N with $\text{ord}_r(N) > n^2$.

Proofs here. Lemma 12 and Theorem 3.

Precise references. The lcm bound is Farhi, Corollary 3, arXiv:0906.2295v2, p. 4 [Far09]. The original AKS search lemma is Lemma 4.3, p. 785 [AKS04]; use its corrected proof in the 2019 erratum, pp. 317–318 [AKS19]. Our constant 2 and integer n are covered by the explicit calculation in Section 8.

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