

## Lecture 18: Pseudorandom Functions

# Pseudo-random Functions (PRF)

- Let  $\mathcal{G}_{m,n,k} = \{g_1, g_2, \dots, g_{2^k}\}$  be a set of functions such that each  $g_i: \{0, 1\}^m \rightarrow \{0, 1\}^n$
- This set of functions  $\mathcal{G}_{m,n,k}$  is called a pseudo-random function if the following holds.

Suppose we pick  $g \xleftarrow{\$} \mathcal{G}_{m,n,k}$ . Let  $x_1, \dots, x_t \in \{0, 1\}^m$  be distinct inputs. Given  $(x_1, g(x_1)), \dots, (x_{t-1}, g(x_{t-1}))$  for any computationally bounded party the value  $g(x_t)$  appears to be uniformly random over  $\{0, 1\}^n$

# Secret-key Encryption using Pseudo-Random Functions

Before we construct a PRF, let us consider the following secret-key encryption scheme.

- ❶  $\text{Gen}()$ : Return  $\text{sk} = \text{id} \xleftarrow{\$} \{1, \dots, 2^k\}$
- ❷  $\text{Enc}_{\text{id}}(m)$ : Pick a random  $r \xleftarrow{\$} \{0, 1\}^m$ . Return  $(m \oplus g_{\text{id}}(r), r)$ , where  $m \in \{0, 1\}^n$ .
- ❸  $\text{Dec}_{\text{id}}(\tilde{c}, \tilde{r})$ : Return  $\tilde{c} \oplus g_{\text{id}}(\tilde{r})$ .

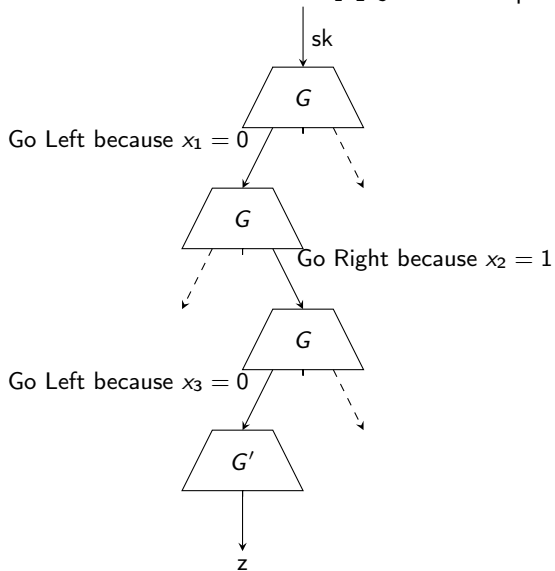
**Features.** Suppose the messages  $m_1, \dots, m_u$  are encrypted as the cipher-texts  $(c_1, r_1), \dots, (c_u, r_u)$ .

- As long as the  $r_1, \dots, r_u$  are all distinct, each one-time pad  $g_{\text{id}}(r_1), \dots, g_{\text{id}}(r_u)$  appear uniform and independent of others to computationally bounded adversaries. So, this encryption scheme is secure against computationally bounded adversaries!
- The probability that any two of the randomness in  $r_1, \dots, r_u$  are not distinct is very small (We shall prove this later as “Birthday Paradox”)
- This scheme is a “state-less” encryption scheme. Alice and Bob do not need to remember any private state (except the secret-key  $\text{sk}$ )!

- We shall consider the construction of Goldreich-Goldwasser-Micali (GGM) construction.
- Let  $G: \{0, 1\}^k \rightarrow \{0, 1\}^{2k}$  be a PRG. We define  $G(x) = (G_0(x), G_1(x))$ , where  $G_0, G_1: \{0, 1\}^k \rightarrow \{0, 1\}^k$
- Let  $G': \{0, 1\}^k \rightarrow \{0, 1\}^n$  be a PRG
- We define  $g_{\text{id}}(x_1 x_2 \dots x_m)$  as follows

$$G' (G_{x_m}(\dots G_{x_2}(G_{x_1}(\text{id}))\dots))$$

Consider the execution for  $x = x_1x_2x_3 = 010$ . Output  $z$  is computed as follows.



We give the pseudocode of algorithms to construct PRG and PRF using a OWP  $f: \{0, 1\}^{k/2} \rightarrow \{0, 1\}^{k/2}$

- Suppose  $f: \{0, 1\}^{k/2} \rightarrow \{0, 1\}^{k/2}$  is a OWP
- We provide the pseudocode of a PRG  $G: \{0, 1\}^k \rightarrow \{0, 1\}^t$ , for any integer  $t$ , using the one-bit extension PRG construction of Goldreich-Levin hardcore predicate construction. Given input  $s \in \{0, 1\}^k$ , it outputs  $G(s)$ .

$G(k, t, s)$ :

- 1 Interpret  $s = (r, x)$ , where  $r, x \in \{0, 1\}^{k/2}$
- 2 Initialize bits = [ ] (i.e., an empty list)
- 3 Initialize  $z = x$
- 4 For  $i = 1$  to  $t$ :
  - 1 bits.append( $\langle r, z \rangle$ ), here  $\langle \cdot, \cdot \rangle$  is the inner-product
  - 2  $z = f(z)$
- 5 Return bits

- We provide the pseudocode of the PRF  
 $g_{id}: \{0, 1\}^m \rightarrow \{0, 1\}^n$ , where  $id \in \{0, 1\}^k$ , using the GGM construction. Given input  $x \in \{0, 1\}^m$ , it outputs  $g_{id}(x)$ .

$g(m, n, k, id, x)$ :

- 1 Interpret  $x = x_1 x_2 \dots x_m$ , where  $x_1, \dots, x_m \in \{0, 1\}$
- 2 Initialize  $inp = id$
- 3 For  $i = 1$  to  $m$ :
  - 1 Let  $y = G(k, 2k, inp)$
  - 2 If  $x_i = 0$ , then  $inp$  is the first  $k$  bits of  $y$ . Otherwise (if  $x_i = 1$ ),  $inp$  is the last  $k$  bits of  $y$ .
- 4 Return  $G(k, n, inp)$

# Naor-Reingold PRF

- This function evaluation is parallelizable, and its security is based on the “Decisional Diffie-Hellman problem” (DDH)
- Let  $p$  and  $\ell$  be prime numbers such that  $\ell$  divides  $(p - 1)$ .
- $g \in F_p^*$  generate a subgroup of order  $\ell$
- Naor-Reingold PRF are functions  $\{0, 1\}^n \rightarrow F_p^*$  defined below:

$$f_a(x) := g^{a_0 \cdot a_1^{x_1} \cdot a_2^{x_2} \cdots a_n^{x_n}},$$

where

$$a = a_0 a_1 \cdots a_n \in (F_\ell)^{n+1}$$

$$x = x_1 x_2 \cdots x_n \in \{0, 1\}^n.$$

- **Note:** For an additive group like an elliptic curve, the definition of the function is

$$f_a(x) := (a_0 \cdot a_1^{x_1} \cdot a_2^{x_2} \cdots a_n^{x_n}) \cdot G,$$

where the subgroup generated by  $G$  has order  $\ell$ .