

Lecture 07: Graph Representation

- We shall develop a new graph representation to argue the privacy and correctness of cryptographic schemes
 - “correctness and privacy” together is referred to as “security”
- As a representative application of this notation, we shall analyze private-key Encryption schemes using graphs

Assumption about Private-key Encryption Schemes

For simplicity of proof and clarity of intuition, we shall consider the class of all private-key encryption algorithms with the following restrictions

- 1 The key-generation algorithm Gen outputs a secret key sampled uniformly at random from the set $\mathcal{K}$
- 2 The encryption algorithm $\text{Enc}_{sk}(m)$ is deterministic

I want to emphasize that with a bit of effort, these *restrictions* can be removed

Graph of Private-key Encryption

Suppose $(\text{Gen}, \text{Enc}, \text{Dec})$ is a private-key encryption scheme that satisfies the two restrictions we mentioned earlier. We construct the following bipartite graph

- The left partite set is the set of all messages $\mathcal{M}$
- The right partite set is the set of all cipher-texts $\mathcal{C}$
- Given a message $m \in \mathcal{M}$ and a cipher-text $c \in \mathcal{C}$, we add an edge (m, c) labeled sk , if we have $c = \text{Enc}_{sk}(m)$

This is the graph corresponding to the encryption scheme $(\text{Gen}, \text{Enc}, \text{Dec})$

Intuition. The edge labeled sk witnesses the fact that the message m is encrypted to the cipher-text c . Or, we write this as $m \xrightarrow{sk} c$. We emphasize that there might be more than one secret key that witnesses the fact that the message m is encrypted to the cipher-text c . Let $\text{wt}(m, c)$ represent the number of secret keys sk such that sk witnesses the fact that c is an encryption of m

Describing Private-key Encryption Schemes

- Until now, we have represented a private-key encryption scheme as a triplet of algorithms (Gen, Enc, Dec)
- Henceforth, we can equivalently express them as graphs

Property One: Characterization of Correctness

Theorem

A private-key encryption scheme $(\text{Gen}, \text{Enc}, \text{Dec})$ is incorrect if and only if there are two distinct messages $m, m' \in \mathcal{M}$, a secret key $\text{sk} \in \mathcal{K}$, and a cipher-text $c \in \mathcal{C}$ such that $m \xrightarrow{\text{sk}} c$ and $m' \xrightarrow{\text{sk}} c$.

- Note that if there are two messages m, m' such that $m \xrightarrow{\text{sk}} c$ and $m' \xrightarrow{\text{sk}} c$, then Bob cannot distinguish whether Alice produced the ciphertext c for the message m or m' . Hence, whatever decoding Bob performs, he is bound to be incorrect
- For the other direction, suppose Bob cannot decode the (sk, c) correctly. If there is a unique $m \in \mathcal{M}$ such that $m \xrightarrow{\text{sk}} c$, then Bob can obviously decode correctly. So, there must be two different messages $m, m' \in \mathcal{M}$ such that $m \xrightarrow{\text{sk}} c$ and $m' \xrightarrow{\text{sk}} c$

Property Two: Correct Schemes Cannot Compress I

Theorem

A correct private-key encryption scheme (Gen, Enc, Dec) has $|\mathcal{C}| \geq |\mathcal{M}|$.

- Suppose not. That is, assume we have a correct private-key encryption scheme with $|\mathcal{C}| < |\mathcal{M}|$.
- Fix any secret key $sk \in \mathcal{K}$.
- Suppose $\mathcal{M} = \{m_1, m_2, \dots, m_\alpha\}$. Consider the following maps

$$m_1 \xrightarrow{sk} c_1$$

$$m_2 \xrightarrow{sk} c_2$$

$$\vdots$$

$$m_\alpha \xrightarrow{sk} c_\alpha$$

Property Two: Correct Schemes Cannot Compress II

Note that these mappings exist because, given any sk and m , the encryption algorithm maps to a unique ciphertext.

- Since $|\mathcal{C}| < |\mathcal{M}|$, by pigeon-hole principle there are two distinct messages $m, m' \in \mathcal{M}$ and a cipher text $c \in \mathcal{C}$ such that $m \xrightarrow{sk} c$ and $m' \xrightarrow{sk} c$
- So the scheme is incorrect. Hence contradiction.

Property Three: Characterization of Privacy I

Theorem

A private-key encryption scheme $(\text{Gen}, \text{Enc}, \text{Dec})$ has privacy if and only if for any c and two distinct messages $m, m' \in \mathcal{M}$ we have $\text{wt}(m, c) = \text{wt}(m', c)$.

- For any $m \in \mathcal{M}$ and $c \in \mathcal{C}$, note that we have $\mathbb{P}[\mathbb{C} = c | \mathbb{M} = m] = \text{wt}(m, c) / |\mathcal{K}|$.
- Exercise: Prove that the privacy definition we have studied is equivalent to saying the following
“For any two distinct messages $m, m' \in \mathcal{M}$ and a cipher-text $c \in \mathcal{C}$ we have: $\mathbb{P}[\mathbb{C} = c | \mathbb{M} = m] = \mathbb{P}[\mathbb{C} = c | \mathbb{M} = m']$ ”
- Given this result, we can conclude that a scheme $(\text{Gen}, \text{Enc}, \text{Dec})$ is private if and only if
“For any two distinct messages $m, m' \in \mathcal{M}$ and a cipher-text $c \in \mathcal{C}$ we have: $\text{wt}(m, c) = \text{wt}(m', c)$ ”

Property Three: Characterization of Privacy II

- **Food for thought.** In a private scheme, if there are $m \xrightarrow{\text{sk}} c$, then for all $m' \in \mathcal{M}$ there exists some sk' such that $m' \xrightarrow{\text{sk}'} c$
- **Food for thought.** The size of the set $\mathcal{K}$ need not be divisible by the size of the set $\mathcal{M}$. However, if there is a message m and a ciphertext c such that $\text{wt}(m, c) = w$, then the number of secret keys must satisfy $|\mathcal{K}| \geq w|\mathcal{M}|$ in a correct and private scheme. Why?

Property Four: Correct+Private Schemes need Lots of Keys

Recall that “correct + private” is identical to “secure”

Theorem

A secure (i.e., correct and private) private-key encryption scheme $(\text{Gen}, \text{Enc}, \text{Dec})$ has $|\mathcal{K}| \geq |\mathcal{M}|$

- Suppose not. That is, there is a correct and private scheme with $|\mathcal{K}| < |\mathcal{M}|$.
- Fix a cipher-text $c \in \mathcal{C}$ such that there exists $m \in \mathcal{M}$ and $\text{sk} \in \mathcal{K}$ such that $m \xrightarrow{\text{sk}} c$. Intuitively, we are picking a ciphertext that has a positive probability. For example, we are not picking a ciphertext that is never actually produced.
- Let the message space be $\mathcal{M} = \{m_1, m_2, \dots, m_\alpha\}$

Property Four: Correct+Private Schemes need Lots of Keys II

- Note that, for any $m_i \in \mathcal{M}$ there exists some sk_i such that $m_i \xrightarrow{sk_i} c$ (This is a property of privacy in private-key encryption schemes that was left as an exercise in the previous slide)
- Now, consider the mappings

$$m_1 \xrightarrow{sk_1} c$$

$$m_2 \xrightarrow{sk_2} c$$

$$\vdots$$

$$m_\alpha \xrightarrow{sk_\alpha} c$$

Property Four: Correct+Private Schemes need Lots of Keys

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- Since $|\mathcal{K}| < |\mathcal{M}|$, by pigeon-hole principle, there exists two distinct messages m_i, m_j such that $sk_i = sk_j$ in the above mappings.
- This violates correctness. Hence contradiction.

Optimality of One-time Pad

- Note that any correct private-key encryption scheme must have $|\mathcal{C}| \geq |\mathcal{M}|$ (property two)
- Note that any secure (i.e., correct and private) private-key encryption scheme must have $|\mathcal{K}| \geq |\mathcal{M}|$ (property four)
- One-time pad is a correct and secure scheme that achieves $|\mathcal{K}| = |\mathcal{C}| = |\mathcal{M}|$ (so it is optimal)

Additional Food for Thought

- Recall that Property four states that the “correctness and privacy” of a private-key encryption scheme implies that the size of the set of keys is greater than or equal to the size of the set of messages. For any $\mathcal{M}$, construct a correct but insecure private-key encryption scheme such that $|\mathcal{K}| = 1$! This result shows the necessity of both correctness and privacy in that property.
- Another natural question is: Can we provide such guarantees for private-key encryption schemes that are private but incorrect? The answer is NO. Think of a private-key encryption scheme that is private (but incorrect) and works for any message set $\mathcal{M}$ and has $|\mathcal{K}| = |\mathcal{C}| = 1$!