

Lecture 18: Pseudorandom Functions

Pseudo-random Functions (PRF)

- Let $\mathcal{G}_{m,n,k} = \{g_1, g_2, \dots, g_{2^k}\}$ be a set of functions such that each $g_i: \{0, 1\}^m \rightarrow \{0, 1\}^n$
- This set of functions $\mathcal{G}_{m,n,k}$ is called a pseudo-random function if the following holds.

Suppose we pick $g \xleftarrow{\$} \mathcal{G}_{m,n,k}$. Let $x_1, \dots, x_t \in \{0, 1\}^m$ be distinct inputs. Given $(x_1, g(x_1)), \dots, (x_{t-1}, g(x_{t-1}))$ for any computationally bounded party the value $g(x_t)$ appears to be uniformly random over $\{0, 1\}^n$

Secret-key Encryption using Pseudo-Random Functions

Before we construct a PRF, let us consider the following secret-key encryption scheme.

- ❶ $\text{Gen}()$: Return $\text{sk} = \text{id} \xleftarrow{\$} \{1, \dots, 2^k\}$
- ❷ $\text{Enc}_{\text{id}}(m)$: Pick a random $r \xleftarrow{\$} \{0, 1\}^m$. Return $(m \oplus g_{\text{id}}(r), r)$, where $m \in \{0, 1\}^n$.
- ❸ $\text{Dec}_{\text{id}}(\tilde{c}, \tilde{r})$: Return $\tilde{c} \oplus g_{\text{id}}(\tilde{r})$.

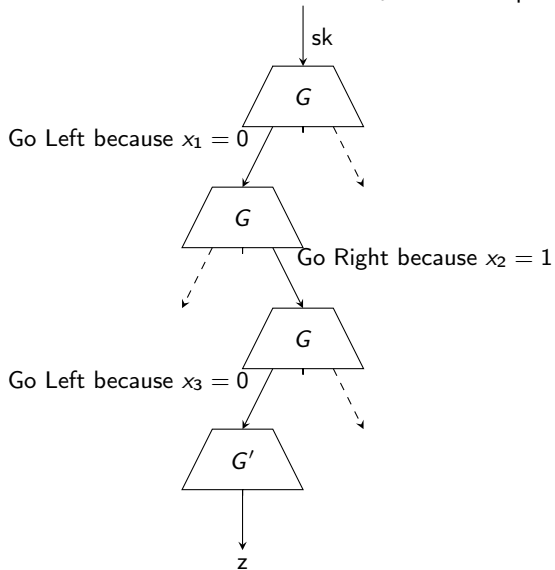
Features. Suppose the messages $m_1, \dots, m_u$ are encrypted as the cipher-texts $(c_1, r_1), \dots, (c_u, r_u)$.

- As long as the $r_1, \dots, r_u$ are all distinct, each one-time pad $g_{\text{id}}(r_1), \dots, g_{\text{id}}(r_u)$ appear uniform and independent of others to computationally bounded adversaries. So, this encryption scheme is secure against computationally bounded adversaries!
- The probability that any two of the randomness in $r_1, \dots, r_u$ are not distinct is very small (We shall prove this later as “Birthday Paradox”)
- This scheme is a “state-less” encryption scheme. Alice and Bob do not need to remember any private state (except the secret-key sk)!

- We shall consider the construction of Goldreich-Goldwasser-Micali (GGM) construction.
- Let $G: \{0, 1\}^k \rightarrow \{0, 1\}^{2k}$ be a PRG. We define $G(x) = (G_0(x), G_1(x))$, where $G_0, G_1: \{0, 1\}^k \rightarrow \{0, 1\}^k$
- Let $G': \{0, 1\}^k \rightarrow \{0, 1\}^n$ be a PRG
- We define $g_{\text{id}}(x_1 x_2 \dots x_m)$ as follows

$$G' (G_{x_m} (\dots G_{x_2} (G_{x_1} (\text{id})) \dots))$$

Consider the execution for $x = x_1x_2x_3 = 010$. Output z is computed as follows.



We give the pseudocode of algorithms to construct PRG and PRF using a OWP $f: \{0, 1\}^{k/2} \rightarrow \{0, 1\}^{k/2}$

- Suppose $f: \{0, 1\}^{k/2} \rightarrow \{0, 1\}^{k/2}$ is a OWP
- We provide the pseudocode of a PRG $G: \{0, 1\}^k \rightarrow \{0, 1\}^t$, for any integer t , using the one-bit extension PRG construction of Goldreich-Levin hardcore predicate construction. Given input $s \in \{0, 1\}^k$, it outputs $G(s)$.

$G(k, t, s)$:

- 1 Interpret $s = (r, x)$, where $r, x \in \{0, 1\}^{k/2}$
- 2 Initialize bits = [] (i.e., an empty list)
- 3 Initialize $z = x$
- 4 For $i = 1$ to t :
 - 1 bits.append($\langle r, z \rangle$), here $\langle \cdot, \cdot \rangle$ is the inner-product
 - 2 $z = f(z)$
- 5 Return bits

- We provide the pseudocode of the PRF
 $g_{id}: \{0, 1\}^m \rightarrow \{0, 1\}^n$, where $id \in \{0, 1\}^k$, using the GGM construction. Given input $x \in \{0, 1\}^m$, it outputs $g_{id}(x)$.

$g(m, n, k, id, x)$:

- 1 Interpret $x = x_1 x_2 \dots x_m$, where $x_1, \dots, x_m \in \{0, 1\}$
- 2 Initialize $inp = id$
- 3 For $i = 1$ to m :
 - 1 Let $y = G(k, 2k, inp)$
 - 2 If $x_i = 0$, then inp is the first k bits of y . Otherwise (if $x_i = 1$), inp is the last k bits of y .
- 4 Return $G(k, n, inp)$

Naor-Reingold PRF

- This function evaluation is parallelizable, and its security is based on the “Decisional Diffie-Hellman problem” (DDH)
- Let p and ℓ be prime numbers such that ℓ divides $(p - 1)$.
- $g \in F_p^*$ generate a subgroup of order ℓ
- Naor-Reingold PRF are functions $\{0, 1\}^n \rightarrow F_p^*$ defined below:

$$f_a(x) := g^{a_0 \cdot a_1^{x_1} \cdot a_2^{x_2} \cdots a_n^{x_n}},$$

where

$$a = a_0 a_1 \cdots a_n \in (F_\ell)^{n+1}$$

$$x = x_1 x_2 \cdots x_n \in \{0, 1\}^n.$$

- **Note:** For an additive group like an elliptic curve, the definition of the function is

$$f_a(x) := (a_0 \cdot a_1^{x_1} \cdot a_2^{x_2} \cdots a_n^{x_n}) \cdot G,$$

where the subgroup generated by G has order ℓ .