Homework 1

1. **Estimating** $(1-x)$ **using** $\exp(\cdot)$ **function.** For $x \in [0,1)$, we know that

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots.$$

   (a) **(5 points)** Prove that $1 - x \leqslant \exp\left(-x - \frac{x^2}{2}\right)$.

   Solution.

(b) **(5 points)** For $x \in [0, 1/2]$, prove that

$$1 - x \geqslant \exp\left(-x - x^2\right).$$

Solution.

2. **Tight Estimations** Provide meaningful upper and lower bounds for the following expressions.

   (a) **(5 points)** $S_n = \sum_{i=1}^{\infty} i^{-\frac{13}{11}}$.
   Hint: Your upper and lower bounds should be constants.
   Solution.

(b) **(10 points)** $A_n = n!$.
Hint: You may want to start by upper and lower bounding $S_n = \sum_{i=1}^{n} \ln i$.
Solution.

(c) **(10 points)** $B_n = \binom{2n}{n}$.
Hint: Note that $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$.
Solution.

3. **Understanding Joint Distribution.** Ten balls are to be tossed into five bins numbered $\{1, 2, 3, 4, 5\}$. Each ball is thrown into a bin uniformly and independently into the bins. For $i \in \{1, 2, 3, 4, 5\}$, let X_i represent the number of balls that fall into bin i.

   (a) **(5 points)** Find the (marginal) distribution of X_1 and compute its expected value.
   Solution.

   (b) **(3 points)** Find the expected value of $X_1 + X_2 + X_3$.
   Solution.

(c) **(7 points)** Find $\Pr[[X_1 = 4 | X_1 + X_2 + X_3 = 7]]$.
Solution.

4. **Sending one bit.**

   Alice intends to send a bit $b \in \{0, 1\}$ to Bob. When Alice sends the bit, it goes through a series of n relays before reaching Bob. Each relay flips the received bit independently with probability p before forwarding that bit to the next relay.

   (a) **(5 points)** Show that Bob will receive the correct bit with probability

   $$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} p^{2k} \cdot (1-p)^{n-2k}.$$

   Hint: Be careful that Alice could be sending either 0 or 1.

   Solution.

(b) **(5 points)** Let us consider an alternative way to calculate this probability. We say that the relay has *bias* q if the probability it flips the bit is $(1-q)/2$. The bias q is a real number between -1 and $+1$. Show that sending a bit through two relays with bias q_1 and q_2 is equivalent to sending a bit through a single relay with bias $q_1 \cdot q_2$.

Solution.

(c) **(5 points)** Prove that the probability you receive the correct bit when it passes through n relays is

$$\frac{1+(1-2p)^n}{2}.$$

Solution.

5. **An Useful Estimate.**

   For an integers n and t satisfying $0 \leqslant t \leqslant n/2$, define

$$P_n(t) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{t}{n}\right)$$

   We will estimate the above expression. (*Remark*: You shall see the usefulness of this estimation in the topic "Birthday Bound" that we shall cover in the forthcoming lectures.)

   (a) **(13 points)** Show that

$$\exp\left(-\frac{t^2}{2n} - \frac{t}{2n} - \frac{\Theta\left(t^3\right)}{6n^2}\right) \geqslant P_n(t) \geqslant \exp\left(-\frac{t^2}{2n} - \frac{t}{2n} - \frac{\Theta\left(t^3\right)}{3n^2}\right).$$

   Solution.

(b) **(2 points)** When $t = \sqrt{2cn}$, where c is a positive constant, the expression above is

$$P_n(t) = \exp\left(\, -c - \ \Theta\left(1/\sqrt{n}\right)\, \right).$$

Solution.