Lecture 01: Mathematical Basics (Summations)
What I am Assuming

I am assuming that you know asymptotic notations. For example, the big-O, little-O notations.
Let us try to write a closed form expression for the following summation

\[ S = \sum_{i=1}^{n} 1 \]

It is trivial to see that \( S = n \)
Now, let us try to write a closed form expression for the following summation

\[ S = \sum_{i=1}^{n} i \]

We can prove that \( S = \frac{n(n+1)}{2} \)

How do you prove this statement? (Use Induction? Use the formula for the Sum of an Arithmetic Progression?)

Using Asymptotic Notation, we can say that \( S = \frac{n^2}{2} + o(n^2) \)
Now, let us try to write a closed form expression for the following summation

\[ S = \sum_{i=1}^{n} i^2 \]

We can prove that \( S = \frac{n(n+1)(2n+1)}{6} \)

- Why is the expression on the right an integer? (Prove by induction that 6 divides \( n(n+1)(2n+1) \) for all positive integer \( n \))
- How do you prove this statement? (Use Induction?)

Using Asymptotic Notation, we can say that \( S = \frac{n^3}{3} + o(n^3) \)
Do we see a pattern here?

Conjecture: For $k \geq 1$, we have $\sum_{i=1}^{n} i^{k-1} = \frac{n^k}{k} + o(n^k)$.

How do we prove this statement?
Let $f$ be an increasing function

For example, $f(x) = x^{k-1}$ is an increasing function for $k > 1$ and $x \geq 0$
Estimating Summations by Integration II

Basics
Observation: “Blue area under the curve” is smaller than the “Shaded area of the rectangle”

- Blue area under the curve is:

\[ \int_{x-1}^{x} f(t) \, dt \]

- Shaded area of the rectangle is:

\[ f(x) \]

So, we have the inequality:

\[ \int_{x-1}^{x} f(t) \, dt \leq f(x) \]

Summing both side from \( x = 1 \) to \( x = n \), we get

\[ \sum_{x=1}^{n} \int_{x-1}^{x} f(t) \, dt \leq \sum_{x=1}^{n} f(x) \]
The left-hand side of the inequality is

$$\int_0^1 f(t) \, dt + \int_1^2 f(t) \, dt + \cdots + \int_{n-1}^n f(t) \, dt = \int_0^n f(t) \, dt$$

So, for an increasing $f$, we have the following lower bound.

$$\int_0^n f(t) \, dt \leq \sum_{x=1}^n f(x) \tag{1}$$
Now, we will upper bound the summation expression. Consider the figure below.
Observation: “Blue area under the curve” is greater than the “Shaded area of the rectangle”

So, we have the inequality:

\[ \int_{x-1}^{x} f(t) \, dt \geq f(x - 1) \]

Now we sum the above inequality from \( x = 2 \) to \( x = n + 1 \)

We get

\[ \int_{1}^{2} f(t) \, dt + \int_{2}^{3} f(t) \, dt + \cdots + \int_{n}^{n+1} f(t) \, dt \geq f(1) + f(2) + \cdots + f(n) \]

So, for an increasing \( f \), we get the following upper bound

\[ \int_{1}^{n+1} f(t) \, dt \geq \sum_{x=1}^{n} f(x) \]  \hspace{1cm} (2)
**Theorem**

For an increasing function $f$, we have

$$
\int_0^n f(t) \, dt \leq \sum_{x=1}^{n} f(x) \leq \int_1^{n+1} f(t) \, dt
$$

**Exercise:**

- Use this theorem to prove that $\sum_{i=1}^{n} i^{k-1} = \frac{n^k}{k} + o(n^k)$, for $k \geq 1$

- Consider the function $f(x) = \frac{1}{x}$ to find upper and lower bounds for the sum $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ using the approach used to prove Theorem 1
Differentiation and Integration

- Differentiation: $f'(x)$ represents the slope of the curve $y = f(x)$ at $x$
- Integration: $\int_a^b f(t) \, dt$ represents the area under the curve $y = f(x)$ between $x = a$ and $x = b$
- Increasing function:
  - Observation: The slope an increasing function is positive
  - So, “$f$ is increasing at $x$” is equivalent to “$f'(x) > 0$,” i.e. $f'$ is positive at $x$
- Suppose we want to mathematically write “Slope of a function $f$ is increasing”
  - The “slope of a function $f$” is the function “$f''$”
  - So, the statement “slope of a function $f$ is increasing” is equivalent to “$(f')' \equiv f''$ is positive”
Definition (Concave Upwards Function)

A function $f$ is **concave upwards** in the interval $[a, b]$ if $f''$ is positive in the interval $[a, b]$.

- Example of functions that concave upwards: $x^2$, $\exp(x)$, $1/x$ (in the interval $(0, \infty)$), $x \log x$ (in the interval $(0, \infty)$)

- We emphasize that a “concave upwards” function need not be increasing, for example $f(x) = 1/x$ (for positive $x$) is decreasing
Consider the coordinates \((x - 1, f(x - 1))\) and \((x, f(x))\)

For a concave upwards function, the secant between the two coordinates is always (on or) above the part of the curve \(f\) between the two coordinates.
So, the shaded area of the trapezium is greater than the blue area under the curve.

\[ f(x - 1) \]

\[ f(x) \]
So, we get
\[
\frac{f(x - 1) + f(x)}{2} \geq \int_{x-1}^{x} f(t) \, dt
\]

Now, use this new observation to obtain a better lower bound for the sum \( \sum_{x=1}^{n} f(x) \)

Think: Can you get even tighter bounds?

Additional Reading: Read on the “trapezoidal rule”
Consider the objective of estimating \( n! \) using elementary functions.

Note that one can convert this estimation of products into estimation of sums by taking log. For example,

\[
\ln(n!) = \sum_{i=1}^{n} \ln(i).
\]

Now, one can tightly upper and lower bound the expression \( \sum_{i=1}^{n} \ln(i) \). Use the techniques in the previous slides to obtain meaningful upper and lower bounds of this expression. Suppose

\[
L_n \leq \sum_{i=1}^{n} \ln(i) \leq U_n.
\]

Therefore, one concludes that

\[
\exp(L_n) \leq n! \leq \exp(U_n).
\]
Consider the objective of estimating a fraction $A_n/B_n$

Suppose we have $A_n \leq U_n$ and $L'_n \leq B_n$. Note that

$$\frac{1}{B_n} \leq \frac{1}{L'_n}.$$ 

Note that multiplying with $A_n \leq U_n$, one gets that

$$\frac{A_n}{B_n} \leq \frac{U_n}{L'_n}.$$ 

To summarize, upper-bounding a fraction involves upper-bounding the numerator and lower-bounding the denominator.

Analogously, if $L_n \leq A_n$ and $B_n \leq U'_n$, then we get $\frac{L_n}{U'_n} \leq \frac{A_n}{B_n}$.

**Food for thought.** Provide meaningful upper and lower bound the expression $(\frac{2n}{n}) := \frac{(2n)!}{(n!)^2}$.