Homework 4

1. **An Example of Extended GCD Algortihm (20 points).** Recall that the extended GCD algorithm takes as input two integers a, b and returns a triple (g, α, β), such that

$$g = \gcd(a, b), \text{ and } g = \alpha \cdot a + \beta \cdot b.$$

Here $+$ and $\cdot$ are integer addition and multiplication operations, respectively.

Find (g, α, β) when $a = 2022, b = 125$. You must show all your work.

Solution.

2. **(20 points).** Suppose we have a cryptographic protocol P_n that is implemented using αn^2 CPU instructions, where α is some positive constant. We expect the protocol to be broken with $\beta 2^{n/10}$ CPU instructions.

   Suppose, today, everyone in the world uses the primitive P_n using $n = n_0$, a constant value such that even if the entire computing resources of the world were put together for 8 years we cannot compute $\beta 2^{n_0/10}$ CPU instructions.

   Assume Moore's law holds. That is, every two years, the amount of CPU instructions a CPU can run per second doubles.

   (a) (5 points) Assuming Moore's law, how much faster will be the CPUs 8 years into the future as compared to the CPUs now?
   Solution.

   (b) (5 points) At the end of 8 years, what choice of n_1 will ensure that setting $n = n_1$ will ensure that the protocol P_n for $n = n_1$ cannot be broken for another 8 years?
   Solution.

(c) (5 points) What will be the run-time of the protocol P_n using $n = n_1$ on the new computers as compared to the run-time of the protocol P_n using $n = n_0$ on today's computers?
Solution.

(d) (5 points) What will be the run-time of the protocol P_n using $n = n_1$ on today's computers as compared to the run-time of the protocol P_n using $n = n_0$ on today's computers?
Solution.

(*Remark*: This problem explains why we demand that our cryptographic algorithms run in polynomial time and it is exponentially difficult for the adversaries to break the cryptographic protocols.)

3. **Finding Inverse Using Extended GCD Algorithm (20 points).** In this problem we shall work over the group $(\mathbb{Z}_{257}^*, \times)$. Note that 257 is a prime. The multiplication operation $\times$ is "integer multiplication $\mod 257$."

   Use the Extended GCD algorithm to find the multiplicative inverse of 55 in the group $(\mathbb{Z}_{257}^*, \times)$. You must show all your work.

   Solution.

4. **Another Application of Extended GCD Algorithm (20 points).** Use the Extended GCD algorithm to find $x \in \{0, 1, 2, \dots, 2235\}$ that satisfies the following two equations.

$$\begin{aligned} x &= 2 \mod 52 \\ x &= 3 \mod 43 \end{aligned}$$

Note that 43 is a prime, but 52 is not a prime. However, we have the guarantee that 52 and 43 are relatively prime, that is, $\gcd(52, 43) = 1$. Also note that the number $2235 = 52 \cdot 43 - 1$. You must show all your work.

Solution.

5. **Square Root of an Element (20 points).** Let p be a prime such that $p = 3 \mod 4$. For example, $p \in \{3, 7, 11, 19 \dots\}$.

   We say that x is a square-root of a in the group $(\mathbb{Z}_p^*, \times)$ if $x^2 = a \mod p$. We say that $a \in \mathbb{Z}_p^*$ is a quadratic residue if $a = x^2 \mod p$ for some $x \in \mathbb{Z}_p^*$. Prove that if $a \in \mathbb{Z}_p^*$ is a quadratic residue then $a^{(p+1)/4}$ is a square-root of a.

   (Remark: This statement is only true if we assume that a is a quadratic residue. For example, when $p = 7$, 3 is not a quadratic residue, so $3^{(7+1)/4}$ is not a square root of 3.)

   Solution.

Collaborators :