Homework 1

1. **Estimating logarithm function (15 points).** For $x \in [0, 1)$, we shall use the identity that

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots.$$ 

(a) **(5 points)** For $x \in [0, 1)$, prove that $\ln(1 - x) \leq -x - \frac{x^2}{2}$.

**Solution.**
(b) **(10 points)** For \( x \in [0, 1/2] \), prove that
\[
\ln(1 - x) \geq -x - \frac{x^2}{2 \cdot 2^0} - \frac{x^2}{2 \cdot 2^1} - \frac{x^2}{2 \cdot 2^2} - \cdots = -x - x^2.
\]

**Solution.**
2. **Tight Estimations (25 points)** Provide meaningful upper-bounds and lower-bounds for the following expressions.

   (a) (10 points) \( S_n = \sum_{i=1}^{n} \ln i, \)
   
   **Solution.**
(b) (5 points) $A_n = n!$

Solution.
(c) (10 points) \( B_n = \binom{2n}{n} = \frac{(2n)!}{(n!)^2} \)

Solution.
3. Understanding Joint Distribution (15 points) Recall that in the lectures we considered the joint distribution \((T, B)\) over the sample space \(\{4, 5, \ldots, 10\} \times \{T, F\}\), where \(T\) represents the time I wake up in the morning, and \(B\) represents whether I have breakfast or not. The following table summarizes the joint probability distribution. (Note that this table may not be the same as the one you saw in lecture.)

<table>
<thead>
<tr>
<th>(t)</th>
<th>(b)</th>
<th>(P[T = t, B = b])</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>T</td>
<td>0.03</td>
</tr>
<tr>
<td>4</td>
<td>F</td>
<td>0.06</td>
</tr>
<tr>
<td>5</td>
<td>T</td>
<td>0.05</td>
</tr>
<tr>
<td>5</td>
<td>F</td>
<td>0.05</td>
</tr>
<tr>
<td>6</td>
<td>T</td>
<td>0.08</td>
</tr>
<tr>
<td>6</td>
<td>F</td>
<td>0.04</td>
</tr>
<tr>
<td>7</td>
<td>T</td>
<td>0.14</td>
</tr>
<tr>
<td>7</td>
<td>F</td>
<td>0.03</td>
</tr>
<tr>
<td>8</td>
<td>T</td>
<td>0.13</td>
</tr>
<tr>
<td>8</td>
<td>F</td>
<td>0.12</td>
</tr>
<tr>
<td>9</td>
<td>T</td>
<td>0.06</td>
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<tr>
<td>9</td>
<td>F</td>
<td>0.10</td>
</tr>
<tr>
<td>10</td>
<td>T</td>
<td>0.02</td>
</tr>
<tr>
<td>10</td>
<td>F</td>
<td>0.09</td>
</tr>
</tbody>
</table>

Calculate the following probabilities.

(a) (5 points) Calculate the probability that I wake up at 8 a.m. or earlier, but do not have breakfast. That is, calculate \(P[T \leq 8, B = F]\).

Solution.
(b) \textbf{(5 points)} Calculate the probability that I wake up at 8 a.m. or earlier. That is, calculate $P[T \leq 8]$.

\textbf{Solution.}
(c) (5 points) Calculate the probability that I skip breakfast conditioned on the fact that I woke up at 8 a.m. or earlier. That is, compute $\Pr[B = F \mid T \leq 8]$.

Solution.
4. **Random Walk (20 points).** There is a frog sitting at the origin \((0, 0)\) in the first quadrant of a two-dimensional Cartesian plane. The frog first jumps uniformly at random along the X-axis to some point \((X, 0)\), where \(X \in \{1, 2, 3, 4, 5, 6\}\). Then, it jumps uniformly at random along the Y-axis to some point \((X, Y)\), where \(Y \in \{1, 2, 3, 4, 5, 6\}\). So \((X, Y)\) represents the final position of the frog after these two jumps. Note that \(X\) and \(Y\) are two independent random variables that are uniformly distributed over their respective sample spaces.

(a) **(5 points)** What is the probability that the frog jumps 5 or more units along the Y-axis. That is, compute \(P[Y \geq 5]\).

**Solution.**
(b) (5 points) What is the probability that the Euclidean distance of the final position of the frog from the origin is at least 7. That is compute $P\left[ \sqrt{X^2 + Y^2} \geq 7 \right]$?

**Solution.**
(c) (10 points) What is the probability that the frog has jumped at least 5 units along X-axis conditioned on the fact that the distance of the final position of the frog from the origin is at least 7? That is, compute \( P \left[ X \geq 5 \mid \sqrt{X^2 + Y^2} \geq 7 \right] \)?

**Solution.**
5. **Coin Tossing Word Problem** (15 points). We have three (independent) coins represented by random variables $C_1, C_2,$ and $C_3$.

(i) The first coin has $P[C_1 = H] = \frac{1}{2}, P[C_1 = T] = \frac{1}{2}$,

(ii) The second coin has $P[C_2 = H] = \frac{3}{4}$ and $P[C_2 = T] = \frac{1}{4}$, and

(iii) The third coin has $P[C_3 = H] = \frac{1}{4}$ and $P[C_3 = T] = \frac{3}{4}$.

Consider the following experiment.

(A) Toss the first coin. Let the outcome of the first coin-toss be $\omega_1$.

(B) If $\omega_1 = H$, then we toss the first coin once again and then toss the third coin once. Otherwise, (i.e., if $\omega_1 = T$) toss the second coin once and then toss the third coin once. Let the two outcomes of this step be represented by $\omega_2$ and $\omega_3$.

(C) Output $(\omega_1, \omega_2, \omega_3)$.

Based on this experiment, compute the probabilities below.

(a) **(5 points)** In the experiment mentioned above, what is the probability that a majority of the three outcomes $(\omega_1, \omega_2, \omega_3)$ are $H$ (head)?

   **Solution.**
(b) \textbf{(10 points)} In the experiment mentioned above, what is the probability that a majority of the three outcomes are $H$, conditioned on the fact that the first outcome was $T$?

\textbf{Solution.}
6. **(10 points)** Use the fact that \( \exp(-x) \approx 1 - x \) (when \( x \) is small) to show

\[
\left( 1 - \frac{1}{n} \right) \left( 1 - \frac{2}{n} \right) \ldots \left( 1 - \frac{t-1}{n} \right) \approx \left( 1 - \frac{(t-1)t}{2n} \right)
\]

when \( t^2/n \) is small.

*(Remark: You shall see the usefulness of this estimation in the topic “Birthday Bound” that we shall cover in the forthcoming lectures.)*

**Solution.**
Collaborators: