Homework 1

1. **Estimating logarithm function.** For $x \in [0,1)$, we shall use the identity that

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots.$$

   (a) **(5 points)** Prove that $\ln(1-x) \leqslant -x - \frac{x^2}{2}$.
   Solution.

(b) **(5 points)** For $x \in [0, 1/2]$, prove that

$$\ln(1-x) \geqslant -x - x^2.$$

Solution.

2. **Tight Estimations** Provide meaningful upper-bounds and lower-bounds for the following expressions.

   (a) **(10 points)** $S_n = \sum_{i=1}^{n} \ln i$.
   Solution.

(b) **(5 points)** $A_n = n!$.
Solution.

(c) **(10 points)** $B_n = \binom{2n}{n} = \frac{(2n)!}{(n!)^2}$.

Solution.

3. **Understanding Joint Distribution.** Recall that in the lectures we considered the joint distribution $(\mathbb{T}, \mathbb{B})$ over the sample space $\{4, 5, \ldots, 11\} \times \{\mathsf{T}, \mathsf{F}\}$, where $\mathbb{T}$ represents the time I wake up in the morning, and $\mathbb{B}$ represents whether I have breakfast or not. The following table summarizes the joint probability distribution.

t	b	$\mathbb{P}[\mathbb{T} = t, \mathbb{B} = b]$
4	T	0.04
4	F	0.02
5	T	0.03
5	F	0.01
6	T	0.20
6	F	0.10
7	T	0.30
7	F	0.05
8	T	0.15
8	F	0
9	T	0.04
9	F	0.04
10	T	0
10	F	0.01
11	T	0
11	F	0.01

Calculate the following probabilities.

(a) **(5 points)** Calculate the probability that I wake up at 9 a.m. or earlier, but do not have breakfast. That is, calculate $\mathbb{P}[\mathbb{T} \leqslant 9, \mathbb{B} = \mathsf{F}]$.

Solution.

(b) **(5 points)** Calculate the probability that I wake up at 9 a.m. or earlier. That is, calculate $\mathbb{P}\left[\mathbb{T} \leqslant 9\right]$.

Solution.

(c) **(5 points)** Calculate the probability that I skip breakfast conditioned on the fact that I woke up at 9 a.m. or earlier. That is, compute $\mathbb{P}\left[\mathbb{B} = \mathsf{F} \mid \mathbb{T} \leqslant 9\right]$.

Solution.

4. **Random Walk.** There is a frog sitting at the origin $(0, 0)$ in the first quadrant of a two-dimensional Cartesian plane. The frog first jumps uniformly at random along the X-axis to some point $(\mathbb{X}, 0)$, where $\mathbb{X} \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. Then, it jumps uniformly at random along the Y-axis to some point $(\mathbb{X}, \mathbb{Y})$, where $\mathbb{Y} \in \{1, 2, 3, 4, 5\}$. So $(\mathbb{X}, \mathbb{Y})$ represents the final position of the frog after these two jumps. Note that $\mathbb{X}$ and $\mathbb{Y}$ are two independent random variables that are uniformly distributed over their respective sample spaces.

   (a) **(5 points)** What is the probability that the frog jumps more than 3 units along the Y-axis. That is, compute $\mathbb{P}[\mathbb{Y} > 3]$.
   Solution.

   (b) **(5 points)** What is the probability that the final position of the frog is within the circle $X^2 + Y^2 = 9$? That is, compute $\mathbb{P}\left[\mathbb{X}^2 + \mathbb{Y}^2 \leqslant 9\right]$.
   Solution.

(c) **(5 points)** What is the probability that the frog has jumped 2 units along X-axis conditioned on the fact that its final position is outside the circle $X^2 + Y^2 = 9$? That is, compute $\mathbb{P}\left[\mathbb{X} = 2 | \mathbb{X}^2 + \mathbb{Y}^2 > 9\right]$.

Solution.

5. **Coin Tossing Word Problem.** We have three (independent) coins represented by random variables $\mathbb{C}_1, \mathbb{C}_2$, and $\mathbb{C}_3$.

   (i) The first coin has $\mathbb{P}[\mathbb{C}_1 = H] = \frac{2}{7}, \mathbb{P}[\mathbb{C}_1 = T] = \frac{5}{7}$,
   (ii) The second coin has $\mathbb{P}[\mathbb{C}_2 = H] = \frac{3}{4}$ and $\mathbb{P}[\mathbb{C}_2 = T] = \frac{1}{4}$, and
   (iii) The third coin has $\mathbb{P}[\mathbb{C}_3 = H] = \frac{2}{5}$ and $\mathbb{P}[\mathbb{C}_3 = T] = \frac{3}{5}$.

   Consider the following experiment.

   (A) Toss the first coin. Let the outcome of the first coin-toss be ω_1.
   (B) If $\omega_1 = H$, then we toss the second coin twice. Otherwise, (i.e., if $\omega_1 = T$) toss the third coin twice. Let the two outcomes of this step be represented by ω_2 and ω_3.
   (C) Output $(\omega_1, \omega_2, \omega_3)$.

   Based on this experiment, compute the probabilities below.

   (a) **(5 points)** In the experiment mentioned above, what is the probability that a majority of the three outcomes $(\omega_1, \omega_2, \omega_3)$ are H (head)?
   Solution.

(b) **(5 points)** In the experiment mentioned above, what is the probability that a majority of the three outcomes are H, conditioned on the fact that the first outcome was T?

Solution.

(c) **(5 points)** In the experiment mentioned above, what is the probability that a majority of the three outcomes are different from the first outcome?

Solution.

6. **An Useful Estimate.**

For an integers n and t satisfying $0 \leqslant t \leqslant n/2$, define

$$P_n(t) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{t}{n}\right)$$

We will estimate the above expression. (*Remark*: You shall see the usefulness of this estimation in the topic "Birthday Paradox" that we shall cover in the forthcoming lectures.)

(a) **(10 points)** Show that

$$\exp\left(-\frac{t^2}{2n} - \frac{t}{2n} - \frac{\Theta\left(t^3\right)}{6n^2}\right) \geqslant P_n(t) \geqslant \exp\left(-\frac{t^2}{2n} - \frac{t}{2n} - \frac{\Theta\left(t^3\right)}{3n^2}\right).$$

Solution.

(b) **(5 points)** When $t = \sqrt{2cn}$, where c is a positive constant, the expression above is

$$P_n(t) = \exp\left(-c - \Theta\left(1/\sqrt{n}\right)\right)$$

Solution.

7. **Jensen's Inequality Proof.(15 points)** In this problem, our objective is to prove the Jensen's inequality using the Lagrange form of the Taylor's Remainder Theorem. These theorems are presented below.

Definition 1 (Convex Functions)**.** *A twice-differentiable function* $g \colon \mathbb{R} \to \mathbb{R}$ *is convex if (and only if)* $g''(x) \geqslant 0$ *for all* $x \in \mathbb{R}$.

Theorem 1 (Jensen's Inequality)**.** *Let* $g \colon \mathbb{R} \to \mathbb{R}$ *be a convex function. Then, for any* $p \in [0,1]$ *and* $q = 1-p$*, the following inequality holds:*

$$g(px+qy) \leqslant p \cdot g(x) + q \cdot g(y).$$

Let $f^{(i)}$ be the i-th derivative of the function $f \colon \mathbb{R} \to \mathbb{R}$.

Theorem 2 (Lagrange form of the Taylor's Remainder Theorem)**.** *For any* $k \in \mathbb{N}$ *and* $(k+1)$*-differentiable function* f *the following result holds. For every* $a, \varepsilon \in \mathbb{R}$, *there exists* $\theta \in (0,1)$ *such that*

$$f(a+\varepsilon) = \left(f(a) + f^{(1)}(a)\frac{\varepsilon}{1!} + f^{(2)}(a)\frac{\varepsilon^2}{2!} + \dots + f^{(k)}(a)\frac{\varepsilon^k}{k!} \right) + f^{(k+1)}(a+\theta\varepsilon)\frac{\varepsilon^{k+1}}{(k+1)!}$$

where the term $R = f^{(k+1)}(a+\theta\varepsilon)\frac{\varepsilon^{k+1}}{(k+1)!}$ *is the Lagrange Remainder.*

Solution.

8. **Bonus Problem: Generalized Jensen's Inequality [Probability].(0 points)** In the context of probability theory, Jensen's Inequality is typically stated in the following form in terms of the expected value.

Theorem 3 (Jensen's Inequality)**.** *Let* $\mathbf{X}$ *be a real-valued random variable. Let* f *be a convex function. Suppose* $f\left(\mathbb{E}[\mathbf{X}]\right)$ *and* $\mathbb{E}[f(\mathbf{X})]$ *are both finite. Then*

$$f\left(\mathbb{E}[\mathbf{X}]\right) \leqslant \mathbb{E}[f(\mathbf{X})].$$

Solution.