Lecture 24: Digital Signatures using RSA
Assumption
Bob wants to receive encrypted messages. So, Bob fixes $n$, the number of bits in the primes he wants to choose. Bob picks two random $n$-bit primes $p$ and $q$. Bob computes $N = p \cdot q$. Bob samples a random $e \in \mathbb{Z}^*_\varphi(N)$. Bob computes $d \in \mathbb{Z}^*_\varphi(N)$ such that $e \cdot d = 1 \mod \varphi(N)$ using the extended GCD algorithm. Bob set $pk = (n, N, e)$ and $trap = d$.

The public-key for Bob $pk$ is broadcast to everyone.

To encrypt a message $m \in \{0, 1\}^{n/2}$, Alice runs the $Enc_{pk}(m)$ algorithm defined as follows. Alice samples $r \in \{0, 1\}^{n/2}$ and computes $c = (r||m)^e \mod N$. The cipher-text is $c$.

After receiving a cipher-text $\tilde{c}$, Bob runs the decryption algorithm $Dec_{pk, trap}(\tilde{c})$. Bob computes $(\tilde{r}, \tilde{m}) = \tilde{c}^d \mod N$. 

RSA Encryption
Correctness. We have seen that this public-key encryption is always correct (relies on the fact that $\gcd(e, \varphi(N)) = 1$)

Security. We have seen that this public-key encryption scheme is secure as long as the randomness $r$ used in every encryption algorithm is distinct against computationally bounded eavesdroppers (relies on the birthday bound and the RSA assumption)
Recall that we have seen that the function $f_e : \mathbb{Z}_N^* \rightarrow \mathbb{Z}_N^*$ defined by $f_e(x) = x^e \mod N$ is a bijection that is efficient to evaluate. We shall abstract this concept as “Evaluation is efficient”

Recall that the inverse function $f_e^{-1} : \mathbb{Z}_N^* \rightarrow \mathbb{Z}_N^*$ is efficient to evaluate given $d$, where $e \cdot d = 1 \mod \varphi(N)$; otherwise, not. We shall abstract this concept as “Inversion is inefficient”

In a public-key encryption we want that the “encryption algorithm is efficient” and “decryption algorithm is inefficient.” So, we used the evaluation of $f_e$ for encryption and the inversion of $f_e$ for decryption.
In a digital signature scheme, the signer publishes a public-key \( pk \) and keeps a trapdoor trap with herself. 

Later, if the signer wants to endorse a message \( m \) then she uses an algorithm \( \text{Sign}_{pk,\text{trap}}(m) \) to generate a signature \( \sigma \).

Everyone should be able to verify that “the publisher of the public-key \( pk \) endorses the message \( \tilde{m} \) using the signature \( \tilde{\sigma} \)” by running the verification algorithm \( \text{Ver}_{pk}(\tilde{m},\tilde{\sigma}) \).

An adversary who sees the public-key \( pk \) and a few message-signature pairs \((m_1,\sigma_1),(m_2,\sigma_2),\ldots,(m_k,\sigma_k)\) cannot forge a valid signature \( \sigma' \) on a new message \( m' \).
First observe that we want “verification to be efficient” and “signing to be inefficient”

So, using the ideas in the “abstraction slide,” the idea is to use “evaluation of $f_e$” for verification and “inversion of $f_e$” for signing
Alice decides to endorse messages using $n$-bit primes. Alice picks two random $n$-bit prime numbers $p, q$. Alice computes $N = p \cdot q$ and samples a random $e \in \mathbb{Z}^*_\varphi(N)$. Alice computes $d$ such that $e \cdot d = 1 \mod \varphi(N)$. Alice sets $pk = (n, N, e)$ and $trap = d$.

To sign a message $m \in \{0, 1\}^n$, Alice runs $Sign_{pk, trap}(m)$ defined as follows. Compute $\sigma = m^d \mod N$.

To verify a message-signature pair $(\tilde{m}, \tilde{\sigma})$, Bob runs the verification algorithm $Ver_{pub}(\tilde{m}, \tilde{\sigma})$ defined as follows. Output $\tilde{m} == \tilde{\sigma}^e \mod N$. 
THIS SCHEME IS INSECURE!
Pick any $\sigma' \in \mathbb{Z}_N^*$
Compute $m' = (\sigma')^e \mod N$
Note that this is an efficient attack
Note that we did not even need to see any other message-signature pairs
Although, we do not have any “control” over the message. It is a valid forgery nonetheless
We want to use the fact that in the previous forgery attack, the adversary did not have any control over the message that was being signed.

So, here is the idea underlying the fix. We shall pick a random $r \in \{0, 1\}^{n/2}$ and include $r$ in the public-key $pk$. To sign a message $m \in \{0, 1\}^{n/2}$, we compute $(r \parallel m)$ and compute the signature $\sigma = (r \parallel m)^d \pmod{N}$. To verify a message-signature pair $(\tilde{m}, \tilde{\sigma})$, Bob (the verifier) checks $(r, \tilde{m}) = (\tilde{\sigma})^e \pmod{N}$.

The formal scheme is presented next.
Fixing our original construction

\[ \text{Gen}(1^n) : \]
- Pick random \( n \)-bit primes \( p \) and \( q \).
- Compute \( N \) and \( \varphi(N) \).
- Sample \( e \in \mathbb{Z}^*_\varphi(N) \).
- Compute \( d \) such that \( e \cdot d \equiv 1 \pmod{\varphi(N)} \).
- Sample random \( r \in \{0, 1\}^{n/2} \).
- Return \( pk = (n, N, e, r) \) and \( \text{trap} = d \)
Fixing our original construction

Sign\textsubscript{pk,trap}(m):
- Return \((r \parallel m)^d \mod N\)

Ver\textsubscript{pk}(\tilde{m}, \tilde{\sigma}):
- Return \((r \parallel \tilde{m}) =\tilde{\sigma}^e \mod N\)

In the next lecture we shall learn how to sign arbitrary-length messages \(m \in \{0, 1\}^*\).