

Resilience of Inner-Product Masking Scheme against Hamming Weight Leakage

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Abstract

Additive masking is a widely used countermeasure against side-channel attacks in which a secret is additively split into multiple random shares. However, over binary fields, the number of 1’s in the binary representation (i.e., the Hamming weight) of the shares reveals information about the original secret. Inner product masking scheme has been proposed as a promising alternative that is secure against such information leakage.

In this work, we establish that inner product masking over a binary extension field is provably secure against Hamming weight leakage, and it translates into security against arbitrary symmetric function leakage from the shares. In addition, we present an efficiently computable score function that quantifies its security against leakages, enabling users to test and certify its security. Finally, we derive a relationship between the leakage resilience of inner-product masking and additive masking over an arbitrary field; they are at least as secure as additive masking.

Our approach is Fourier-analytic and involves estimating spectral norms of the Hamming slice by studying Krawtchouk polynomials.

Keywords: Local leakage resilience, additive masking, inner product masking, Hamming weight leakage, Fourier analysis, Krawtchouk polynomials.

1 Introduction

Side-channel attacks have repeatedly compromised cryptographic secrets by exploiting observable side effects of a device’s physical operation. *Masking* [CJRR99, GP99] is a widely prevalent and effective countermeasure; it splits the secret into shares, relying on the noise in measurements from individual shares to accumulate quickly and obscure the secret.

Additive masking, a prominent masking strategy, splits a secret s into random (s_1, s_2, \dots, s_n) shares satisfying $s_1 + s_2 + \dots + s_n = s$. The security provided by this scheme has its limitations; even with noisy measurements, a carefully chosen attack can still reveal partial information about the secret. For example, over any finite field, single-bit probes into the devices storing the shares reveal information about the secret [MNP⁺21, AMN⁺21, MNP⁺22, MNPY24]. *Over binary extension fields, even power analysis and timing attacks* [Koc96, KJJ99] *that reveal only the number of set bits of shares (a.k.a., their Hamming weight) compromise this masking – the secret’s Hamming weight and the sum of all the leaked Hamming weights always have the same parity.* However, over exceptionally specialized and rare Mersenne-prime fields, weak security guarantees are known against such attacks [FMM⁺24].

In comparison, *inner product masking* [BFGV12], a mild generalization of additive masking, has demonstrated significant potential for achieving greater security across all finite fields. Such a

scheme is parameterized by its *reconstruction multipliers* $\vec{\beta} \in (F^*)^n$, where F is the ambient finite field, and the corresponding shares (s_1, s_2, \dots, s_n) satisfy $\beta_1 \cdot s_1 + \beta_2 \cdot s_2 + \dots + \beta_n \cdot s_n = s$. Observe that $\vec{\beta} = (1, 1, \dots, 1)$ reduces it to additive masking; however, carefully selecting these multipliers $\vec{\beta}$ can significantly enhance security. For instance, over any field, nearly all multipliers $\vec{\beta}$ offer strong security guarantees when the number of set bits of each share is revealed [FMM+25]. While nearly all multipliers offer strong security, we cannot identify them, nor can we determine the level of security provided by a specific multiplier.

This work contributes to the ongoing investigations into the security of inner product masking and, by extension, provides guidelines for NIST’s ongoing standardization efforts for more secure secret-sharing schemes [BP23].

Summary of our results. As in [FMM+24, FMM+25], we investigate the *local leakage resilience* [BDIR18, BDIR21] of inner-product masking against *Hamming weight leakage* from shares, where the number of set bits in each share is revealed. This is a strong security metric, requiring statistical independence between the secret and the leakage, and it implies other security metrics in the literature, such as statistical bias [DDF14], mutual information [GBTP08], and guessing entropy [SMY09].

In this work, we establish that inner-product masking is *always* as secure as additive masking over any field, thereby ruling out the possibility of “vulnerable” multipliers that could undermine security – a gap left open in the analysis of [FMM+25]. Over binary extension fields, while additive masking is vulnerable, as indicated above, we prove that it is (essentially) the *only* vulnerable scheme; all other inner product masking schemes are secure. We present an efficiently computable *score function* for multipliers, where a higher score indicates greater security of the corresponding inner-product masking. Our score function enables *certifying* the security achieved by a specific multiplier, facilitating efforts similar to those in [DSV14, DFS15, DSM16]. It helps explicitly identify multipliers with strong security, whose existence was proven in [FMM+25].

The results above stem from more general technical findings, also applicable to the local leakage resilience of secret-sharing schemes and polynomial masking [PR11, GM11]. Our approach is Fourier-analytic and relies on spectral-norm estimates of the Hamming slice, which, in turn, require the asymptotic properties of Krawtchouk polynomials [MS77, van98].

1.1 Preliminary Notations

Basic notation. Here $x = a \pm c$ denotes $x \in [a - c, a + c]$, for reals x and a with $c > 0$. $\text{card}(S)$ denotes the cardinality of the set S . Let F_q denote the finite field of order q and $F_q^* := F_q \setminus \{0\}$. Let $\lambda \in \{1, 2, \dots\}$ satisfy $2^{\lambda-1} < q \leq 2^\lambda$, the number of bits needed to represent the finite field elements. Elements of F_{2^λ} are represented as $F_2[Z]/\Pi(Z)$ elements, where $\Pi(Z)$ is a monic irreducible polynomial of degree λ . The *Hamming weight* of $x \in F_{2^\lambda}$, denoted by $\text{wt}(x)$, is the number of 1’s in x ’s representation.

Secret sharing and Masking. Let Add_n denote the *additive secret sharing* among n parties – to share a secret $s \in F$, it provides random shares $(s_1, s_2, \dots, s_n) \in F^n$ conditioned on $s_1 + s_2 + \dots + s_n = s$. The *generalized additive secret sharing* is parameterized by $\vec{\beta} = (\beta_1, \dots, \beta_n) \in (F^*)^n$, where β_i ’s need not to be all distinct; its shares satisfy $\beta_1 \cdot s_1 + \beta_2 \cdot s_2 + \dots + \beta_n \cdot s_n = s$ instead. Note, the additive scheme corresponds to $\vec{\beta} = (1, \dots, 1)$. The additive and generalized additive secret sharing schemes, respectively, are equivalent to additive and inner-product masking.

Leakage resilience. Fix a secret sharing scheme \mathcal{S} . For, leakage functions $\tau_1, \dots, \tau_n: F \rightarrow \{0, 1\}$, let $\vec{\tau}(s)$ denote the joint leakage distribution $(\tau_1(s_1), \dots, \tau_n(s_n))$, where the \mathcal{S} generates the shares of s . For brevity, $\vec{\tau}(U_F)$ will denote the leakage distribution when the secret is chosen uniformly at random from F . The *local leakage resilience* of \mathcal{S} against the leakage attack $\vec{\tau}$ is defined by [BDIR18,BDIR21]:

$$\varepsilon(\mathcal{S}; \vec{\tau}) := \max_{s \in F} \text{SD}(\vec{\tau}(s), \vec{\tau}(U_F)) \quad (1)$$

In local leakage resilience context, the insecurity of $\text{Gen}(\text{Add}_n, (c, \dots, c))$, for $c \in F^*$, is identical to the Add_n 's insecurity; hence, they are *essentially equivalent*.

Our score function. Define $\sigma: \{0, 1, \dots, \lambda\} \rightarrow \mathbb{R}$ as follows^[1]

$$\sigma(w) := \begin{cases} 0, & \text{if } w \in \{0, \lambda\} \\ \frac{1}{2} \log \binom{\lambda}{w} - \frac{\log \lambda}{4}, & \text{otherwise} \end{cases}. \quad (2)$$

Note that this function can be computed efficiently [Bor85,BZ10].

Remark 1 (Behavior of σ). Since $\binom{\lambda}{w} \geq (\lambda/w)^w$, we have $\sigma(w) \geq (w^*/2) \cdot \log(\lambda/w^*) - (1/4) \cdot \log \lambda$, where $w^* = \min\{w, \lambda - w\}$ and $w \in \{1, 2, \dots, \lambda - 1\}$. Consequently, either $\sigma(w) = 0$ or $\sigma(w) \geq (1/4) \cdot \log \lambda$. Asymptotically, by the central limit theorem, we know that $\binom{\lambda}{\lambda/2 \pm x}$ behaves like $\frac{2^\lambda}{\sqrt{\pi \lambda/2}} \cdot \exp\left(-\frac{2x^2}{\lambda}\right)$. So, $\sigma(\lambda/2 \pm x)$ behaves like $\frac{\log 2}{2} \lambda - \frac{1}{2} \log \lambda - \frac{2x^2}{\lambda}$. When w is drawn according to the binomial distribution, the expected value of $\sigma(w)$ is roughly $\frac{\log 2}{2} \lambda$, a consequence of the entropy of the binomial distribution. \triangle

For $\vec{\beta} \in (F^*)^n$ and $\zeta \in F$, we define $\text{Score}: F \times (F^*)^n \rightarrow \mathbb{R}$ as follows.

$$\text{Score}(\zeta; \vec{\beta}) := \sum_{i=1}^n \sigma(\text{wt}(\beta_i \cdot \zeta)). \quad (3)$$

Let $\zeta^* \in F$ be the (unique) element satisfying $\text{wt}(\zeta^*) = \lambda$. For a tuple $\vec{\beta} = (\beta_1, \beta_2, \dots, \beta_n) \in (F^*)^n$, define

$$\mathcal{S}_{\vec{\beta}} := \{ \zeta^* \cdot \beta_1^{-1}, \zeta^* \cdot \beta_2^{-1}, \dots, \zeta^* \cdot \beta_n^{-1} \} \subseteq F^*. \quad (4)$$

Looking ahead, our security characterization will depend on the minimum ‘score’ $\text{Score}(\zeta; \vec{\beta})$ over $\zeta \in \mathcal{S}_{\vec{\beta}}$; insecurity will be small when $\vec{\beta}$ has a *large minimum score*.

1.2 Our Results

Over binary extension fields. Over F_{2^λ} , the attack in the introduction demonstrates the vulnerability of additive secret sharing to Hamming weight leakage. We begin by demonstrating that transitioning to the generalized setting eliminates this vulnerability.

Informal Theorem 1. *For $n \geq 3$ parties, $\text{Gen}(\text{Add}_n, \vec{\beta})$ is $\lambda^{-1/4}$ -insecure, unless $\vec{\beta} = (c, c, \dots, c)$, for $c \in F^*$.*

^[1]For intuition, <https://www.desmos.com/calculator/zerikstukw> graphs the $\sigma(\cdot)$ function for a representative λ .

Except for those inner product masking that are essentially to the additive masking in the local leakage resilience context, everyone else is secure. [Theorem 1](#) presents this result formally.

Next, we characterize the choices of $\vec{\beta} \in (F^*)^n$ that allow us to derive stronger security bounds for $\text{Gen}(\text{Add}_n, \vec{\beta})$, where insecurity decays $\lambda^{-\Theta(n)}$. This characterization builds on the following technical result.

Informal Theorem 2. *For $n \in \{3, 4, \dots\}$ parties and $\vec{\beta} \in (F^*)^n$, $\text{Gen}(\text{Add}_n, \vec{\beta})$ is λ^{-cn} -insecure, where $c \in (0, 1/2]$ satisfies*

$$\min_{\zeta \in \mathcal{S}_{\vec{\beta}}} \text{Score}(\zeta; \vec{\beta}) \geq cn \cdot \log \lambda. \quad (5)$$

and the Score function is defined in [Equation 3](#).

We leave the case $n = 2$ as an open problem for both results. For clarity of presentation, we ignore additive $\log \log \lambda$ terms in the right-hand side of the expression above. See [Theorem 2](#) for the full expression.

Observe that, given $\vec{\beta}$, one can efficiently compute and test whether the minimum score is $\geq cn \log \lambda$. For example, if all elements of $\vec{\beta}$ are identical, we know already that the $\text{Gen}(\text{Add}_n, \vec{\beta})$ is insecure. In this case, the minimum score is 0 and our technical result cannot upper bound the insecurity by a small quantity.

Consequences of our theorems. The following corollaries are immediate:

1. If every element in $\vec{\beta}$ occurs at most m times, then $\text{Gen}(\text{Add}_n, \vec{\beta})$ satisfies [Equation 5](#) with $c = (n - m)/4n$.

This is because for $\zeta \in \mathcal{S}_{\vec{\beta}}$, $\beta_i \zeta$ is either equal to ζ^* or has weight $\in \{1, 2, \dots, \lambda - 1\}$. Since, every element in $\vec{\beta}$ occurs at most m times, at least $(n - m)$ elements in $\{\zeta \beta_1, \dots, \zeta \beta_n\}$ have weight $\in \{1, 2, \dots, \lambda - 1\}$. Therefore, the score is $\geq (n - m) \cdot (1/4) \log \lambda$; whence, the result.

In particular, if all elements of $\vec{\beta}$ are distinct, then $m = 1$ and $c = (n - 1)/4n$. Moreover, when $\vec{\beta}$ is chosen uniformly at random from F^n , then with exponentially high probability, it consists of distinct elements from F^* .

2. A symmetric $f: F \rightarrow \Omega$ function satisfies the constraint “ $\text{wt}(x) = \text{wt}(y)$ implies $f(x) = f(y)$ ” for $x, y \in F$. Let \mathcal{F} denote the set of all symmetric functions. A symmetric local leakage leaks f_1, \dots, f_n from the shares s_1, \dots, s_n , where $f_1, \dots, f_n \in \mathcal{F}$.

Note that given only the $\text{wt}(x)$, one can compute $f(x)$ for a symmetric function; i.e., $x \rightarrow \text{wt}(x) \rightarrow f(x)$ is a Markov chain. Therefore, by the data processing inequality ([\[CT99, Chapter 2\]](#)), any secret sharing scheme that has ε insecurity against the Hamming weight leakage has ε insecurity against every symmetric local leakage.

Moreover, [Theorem 2](#) gives a systematic way to select reconstruction multipliers $\vec{\beta} \in (F^*)^n$. We consider the case when β_3, \dots, β_n are chosen arbitrarily. As long as β_1 and β_2 are carefully chosen, the inner product scheme has high security.

Informal Corollary 1. *Let $\vec{\beta} \in (F^*)^n$ where the Hamming weight of $\beta_1 \zeta^* \beta_2^{-1}$ and $\beta_2 \zeta^* \beta_1^{-1}$ are not close neither 0 or λ . Then, $\text{Gen}(\text{Add}_n, \vec{\beta})$ satisfies [Equation 5](#) for all sufficiently large λ .*

See [Corollary 1](#) for the full statement. We emphasize that [Corollary 1](#) suggests only two multipliers (namely β_1 and β_2) out of n multipliers in $\vec{\beta} \in (F^*)^n$ need to be adjusted to satisfy [Equation 5](#)

and achieve security. [Corollary 1](#) also allows us to extend our results to the security of Shamir’s secret sharing with random evaluation points ([Remark 7](#)). A detailed discussion is deferred to [Section 3.2](#).

Recommendations for $\vec{\beta}$: an illustrative example

Suppose we are given an insecurity budget of 2^{-50} and $n = 10$, a fairly reasonable expectation in practice. We make the following recommendations based on our technical findings.

1. If $\Pi(Z) = Z^\lambda + Z^j + 1$, where j is a small constant (e.g., $j = 1$): Choose $\vec{\beta} = (\beta, 1, \dots, 1) \in (F^*)^n$, where $\beta = Z^i$, where i is the multiple of j closest to $\lambda/2$. If $\lambda \geq 1024$, the leakage resilience of the inner-product masking parameterized by $\vec{\beta}$ is $\leq 2^{-50}$.
2. If $\Pi(Z) = Z^\lambda + Z^j + 1$, where j is comparable to $\lambda/2$ (e.g., $j = \lfloor \lambda/2 \rfloor$), then choose $\vec{\beta} = (\beta, 1, \dots, 1) \in (F^*)^n$, where $\beta = Z^j$. Again for $\lambda \geq 1024$, we will achieve the desired insecurity bound.
3. If $\Pi(Z) = Z^\lambda + R(Z) + 1$, where $R(Z)$ has more than k monomials, then choose $\vec{\beta} = (\beta, 1, \dots, 1) \in (F^*)^n$, where $\beta = Z$. Then again, for $\lambda \geq 2^{100/(k+1)}$, we will achieve the desired insecurity bound. That is, for moderate k (e.g., $k = 9$), this gives the desired security bound for $\lambda \geq 1024$.

[Appendix F](#) provides proofs supporting our recommendation and more generally applicable recommendations.

Over arbitrary fields. Next, we show that generalized additive secret sharing inherits its security from the additive secret sharing, and is therefore *at least as secure as* the additive secret sharing, in the sense that its insecurity is bounded above by a quantity that upper bounds the insecurity of additive secret sharing.

Informal Theorem 3. *Let F_q be a finite field of order q and $\vec{\beta} \in (F_q^*)^n$. Then $\text{Gen}(\text{Add}_n, \vec{\beta})$ is secure against the leakage $\vec{\tau} = (\tau_1, \tau_2, \dots, \tau_n)$ if Add_n is “sufficiently secure” on average over $i \in \{1, 2, \dots, n\}$ against $\vec{\tau}_i = (\tau_i, \tau_i, \dots, \tau_i)$.*

Remark 2 (Technical subtlety in [Theorem 3](#)). Benhamouda et al. [[BDIR18](#)] introduced the quantity called *Fourier proxy* that measures the local leakage resilience ([Equation 1](#)). Proving that this proxy (or its upper bound) is small is a popular technique to demonstrate that the insecurity is small [[BDIR18](#), [BDIR21](#), [MNP⁺21](#), [MPSW21](#), [MNPY24](#), [KK23](#), [Kas24](#), [Ngu24](#)]. We show that the proxy of the inner product masking is at most the upper bound of the proxy for additive masking, induced by triangle inequality, hence the terminology *sufficiently secure*. We *did not* prove that the insecurity of the inner product masking is at most the insecurity of the additive scheme directly. \triangle

This result also implies that one could verify the security of a generalized additive secret sharing against a certain leakage, by testing whether the additive secret sharing (i.e. when all reconstruction multipliers are 1) is *sufficiently secure* against leakage attacks where the same leakage function is applied to every share. Here, the notion of strong security comes from the fact that we need the upper bound of insecurity to be small, not the insecurity itself. See [Theorem 3](#) for the precise statement and exact expression for upper bounds μ_i of insecurity of additive secret sharing.

We emphasize that [Theorem 3](#) applies to *any* finite field. Consequentially, we can also derive security guarantees against Hamming weight leakage for schemes over prime fields, including those based on Mersenne primes, by combining our generalized bound with the results of Faust et al. [\[FMM⁺24\]](#).

1.3 Technical Overview

Our results are best interpreted by considering the number of parties $n \in \{3, 4, \dots\}$ to be a constant and determining the asymptotics of insecurity as a function of the security parameter λ . Consider $\text{Gen}(\text{Add}_n, \vec{\beta})$, for arbitrary $\vec{\beta} \in (F^*)^n$. We illustrate our analysis strategy against the Hamming weight leakage. The analysis relies on spectral-norm estimates of the Hamming slice, which in turn require the asymptotic properties of Krawtchouk polynomials [\[MS77, van98\]](#). Existing techniques, such as bounding the spectral norm of the Hamming slice via polynomial threshold function estimates, do not yield bounds sufficiently tight for our purposes. See [Appendix C.1](#) for further discussion and [Figure 1](#) for a visualization of the gap between spectral weights at each level.

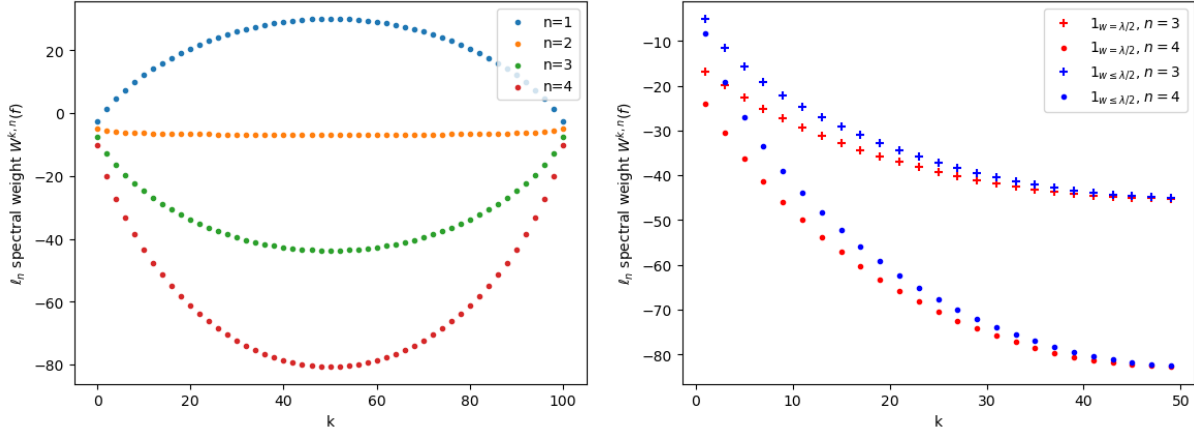


Figure 1: Distribution of ℓ_n -spectral weight at level- k : $W^{k,n}(f) := \sum_{S: \text{wt}(S)=k} |\hat{f}(S)|^n$ where f is a Boolean function over F_{2^λ} .

[Left] x -axis: $k \in \{0, 1, \dots, \lambda\}$. y -axis: $W^{k,n}(\mathbb{1}_{w=\lambda/2})$ (log-scaled) for $n = 1$ (blue), 2 (orange), 3 (green), and 4 (red). Note the change in trend as n increases.

[Right] x -axis: $k \in \{0, 1, \dots, \lambda/2\}$. y -axis: log-scaled $W^{k,n}(\mathbb{1}_{w=\lambda/2})$ (red) and log-scaled $W^{k,n}(\mathbb{1}_{w \leq \lambda/2})$ (blue) for $n = 3$ (+ markers) and 4 (· markers). Note the gap between $\mathbb{1}_{w=\lambda/2}$ (red) and log-scaled $\mathbb{1}_{w \leq \lambda/2}$.

Technical Results

Theorem 1. For $\vec{\beta} \in (F^*)^n \setminus \{(c, \dots, c) : c \in F^*\}$ and $n \in \{3, 4, \dots\}$, we have

$$\varepsilon\left(\text{Gen}(\text{Add}_n, \vec{\beta}) ; \vec{\text{wt}}\right) = \mathcal{O}\left(\lambda^{-1/4}\right).$$

Proof outline. Recall that $\vec{\text{wt}}(s)$, represents the Hamming weight leakage distribution over the sample space $\{0, 1, \dots, \lambda\}^n$ corresponding to a secret $s \in F$. Similarly, $\vec{\text{wt}}(U_F)$ represents the joint Hamming weight distribution when the secret is randomly picked from F . We aim to upper-bound

the statistical distance between these two probability distributions.

$$2 \cdot \text{SD}(\vec{\text{wt}}(s), \vec{\text{wt}}(U_F)) = \sum_{\vec{w} \in \{0,1,\dots,\lambda\}^n} \left| \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; s)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] - \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; U_F)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] \right|$$

Here, $\text{Gen}(\text{Add}_n, \vec{\beta}; s)$ denotes the distribution of shares over F^n for the secret $s \in F$, and $\text{Gen}(\text{Add}_n, \vec{\beta}; U_F)$ the distribution of shares for a uniformly random secret in F . Note that each share s_i is uniformly random over F , irrespective of the secret. Therefore, the (marginal) distribution of the Hamming weight of any share is the binomial distribution $B(\lambda, 1/2)$. By Chernoff bound ([Lemma 1](#)), it is unlikely that the Hamming weight of any share is significantly far from $\lambda/2$. In particular, for a parameter $\tau \in (0, 1/2]$, the Hamming weight of a share being $\geq \lambda^{1/2+\tau}$ far from $\lambda/2$, is at most $2 \cdot \exp(-2\lambda^{2\tau})$. We can control this probability of ‘‘atypical weights’’ by setting $\tau = \frac{\log((n/4) \cdot \log \lambda)}{2 \log \lambda}$ and driving the probability below $\lambda^{-n/2}$.

We define the set of all *typical* leakages $\text{Typical}(n, \tau) \subseteq \{0, 1, \dots, \lambda\}^n$ parameterized by τ :

$$\text{Typical}(n, \tau) := \left\{ \vec{w} : |w_i - \lambda/2| \leq \lambda^{1/2+\tau}, \text{ for all } i \in \{1, 2, \dots, n\} \right\}. \quad (6)$$

The probability that the Hamming weight leakage of any share for secret s or random secret U_F is outside this $\text{Typical}(n, \tau)$ set is (at most) $4n \cdot \exp(-2\lambda^{2\tau})$ by the union bound. So, we conclude that

$$2 \cdot \text{SD}(\vec{\text{wt}}(s), \vec{\text{wt}}(U_F)) = \sum_{\vec{w} \in \text{Typical}(n, \tau)} \left| \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; s)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] - \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; U_F)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] \right| \pm 4n \cdot \lambda^{-n/2}.$$

Therefore, it suffices to estimate the leakage probability restricted to typical leakages $\vec{w} \in \text{Typical}(n, \tau)$.

To this end, we will use Fourier analysis. Using the Poisson summation formula [Lemma 2](#) for the generalized additive secret sharing, we can rewrite the right-hand side expression using the Fourier coefficients.

$$2 \cdot \text{SD}(\vec{\text{wt}}(s), \vec{\text{wt}}(U_F)) = \sum_{\vec{w} \in \text{Typical}(n, \tau)} \left| \sum_{\zeta \in F^*} \left(\prod_{i=1}^n \hat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right) \cdot \chi_1 \left(s \cdot \zeta \cdot \langle \vec{1}, \vec{v} \rangle \right) \right| \pm 4n \cdot \lambda^{-n/2}.$$

Here, $\mathbb{1}_w : F \rightarrow \{0, 1\}$ is the indicator function of the set all elements in F with Hamming weight $w \in \{0, 1, \dots, \lambda\}$. The shares $\vec{v} \in F^n$ is an arbitrary share of the secret $1 \in F$ and $\vec{1} = (1, 1, \dots, 1) \in F^n$. Moreover, $\chi_1 : F \rightarrow \mathbb{C}$ is defined as

$$\chi_1(x) := \exp\left(\frac{2\pi i}{p} \cdot \text{Tr}_{F/F_2}(x)\right).$$

To upper-bound the right-hand side expression, we apply the triangle inequality and conclude:

$$2 \cdot \text{SD}(\vec{\text{wt}}(s), \vec{\text{wt}}(U_F)) \leq \sum_{\vec{w} \in \text{Typical}(n, \tau)} \sum_{\zeta \in F^*} \prod_{i=1}^n \left| \hat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right| + 4n \cdot \lambda^{-n/2}.$$

Now, we need to upper-bound the magnitude of $\widehat{\mathbb{1}}_w(\cdot)$, for typical $w \in \{0, 1, \dots, \lambda\}$. We remind the reader that the right-hand side expression is not small for every $\vec{\beta} \in F^n$. For instance, when $\vec{\beta} = \vec{1}$ we know an attack on the corresponding secret sharing scheme. Thus, the resulting secret sharing scheme must be insecure and the right-hand side expression corresponding to that insecure scheme must be large. So, the right-hand side estimate will be small depending on $\vec{\beta}$; our analysis cannot be transparent to this fact.

To approach this estimation problem, and appreciate our key technical components, let us build some elementary intuition of the magnitude of the Fourier coefficients $\widehat{\mathbb{1}}_w(\cdot)$. We remark that $\widehat{\mathbb{1}}_w(x) = 2^{-\lambda} K_w(\text{wt}(x))$ where $K_w(\cdot)$ is Krawtchouk polynomial, as defined in [Kra01]. Krawtchouk polynomials find extensive applications in many subfields in mathematics, theoretical computer science, and cryptography; see, for example, [van98].

1. First, by symmetry $\widehat{\mathbb{1}}_w(x) = \widehat{\mathbb{1}}_w(y)$, for all $x, y \in F$ satisfying $\text{wt}(x) = \text{wt}(y)$.
2. Next, $\widehat{\mathbb{1}}_w(x) = (-1)^\lambda \cdot \widehat{\mathbb{1}}_w(y)$, for all $x, y \in F$ satisfying $\text{wt}(x) + \text{wt}(y) = \lambda$; because $\mathbb{1}_w$ is a real-valued function.

As we will see, the contributions of $\widehat{\mathbb{1}}_w(x)$, where $\text{wt}(x) \in \{1, 2, \dots, \lambda - 1\}$, is relatively small; *this fact is non-trivial to prove and is one of our technical contributions*. The contributions of $\widehat{\mathbb{1}}_w(0)$ and $\widehat{\mathbb{1}}_w(\zeta^*)$, where $\zeta^* \in F$ is the unique element with $\text{wt}(\zeta^*) = \lambda$, are large. Both of them have magnitude equal to the density of the subset of all weight- w elements of F . However, note that $\widehat{\mathbb{1}}_w(0)$ never occurs on the right-hand side expression, because the right-hand side expression only considers $\zeta \in F^*$ and all β_i are also non-zero. The potential “troublemakers” are those terms where $\widehat{\mathbb{1}}_w(\zeta^*)$ appears.

To account for them, we identify the set of candidate $\zeta \in F$ such that $\zeta \cdot \beta_i = \zeta^*$ for some $i \in \{1, 2, \dots, n\}$.

$$\mathcal{S}_{\vec{\beta}} := \left\{ \zeta^* \cdot \beta_1^{-1}, \zeta^* \cdot \beta_2^{-1}, \dots, \zeta^* \cdot \beta_n^{-1} \right\}.$$

Note that the cardinality of $\mathcal{S}_{\vec{\beta}}$ is the number of distinct elements in $\vec{\beta}$, which is at most n . We split the right-hand side expression depending on whether $\zeta \in \mathcal{S}_{\vec{\beta}}$ or not, then we estimate them separately:

$$\begin{aligned} 2 \cdot \text{SD}(\vec{\text{wt}}(s), \vec{\text{wt}}(U_F)) &\leq \underbrace{\sum_{\vec{w} \in \text{Typical}(n, \tau)} \sum_{\zeta \in \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right|}_{\text{first summand}} \\ &+ \underbrace{\sum_{\vec{w} \in \text{Typical}(n, \tau)} \sum_{\zeta \in F^* \setminus \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right|}_{\text{second summand}} + 4n \cdot \lambda^{-n/2}. \end{aligned}$$

Second summand estimation. We will prove an upper bound on the magnitude $\left| \widehat{\mathbb{1}}_{w_i}(x) \right| \leq B(x)$ as defined in Lemma 4, for all $x \in F \setminus \{0, \zeta^*\}$. Using this bound, we have:

$$\begin{aligned} \text{second summand} &\leq \sum_{\vec{w} \in \text{Typical}(n, \tau)} \sum_{\zeta \in F^* \setminus \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n B(\beta_i \zeta) \\ &= \left(2\lambda^{1/2+\tau} \right)^n \cdot \left(\sum_{\zeta \in F^* \setminus \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n B(\beta_i \zeta) \right). \end{aligned}$$

The last equality substitutes the cardinality of the $\text{Typical}(n, \tau)$ set. Now By using the (generalized) Hölder's inequality ([Lemma 3](#)), we have

$$\text{second summand} \leq \left(2\lambda^{1/2+\tau}\right)^n \cdot \left(\sum_{\zeta \in F \setminus \{0, \zeta^*\}} B(\zeta)^n\right) \quad (7)$$

We show that this n -th norm, for $n \geq 3$, is small. Roughly speaking, [Lemma 4](#) upper bounds the right-hand side expression as follows:

$$\text{second summand} \leq \left(2\lambda^{1/2+\tau}\right)^n \cdot \lambda^{-n+1} = (2\lambda^\tau)^n \cdot \lambda^{-n/2+1}. \quad (8)$$

which is $1/\text{poly}(\lambda)$ for $n \geq 3$; here, the exponent of the polynomial depends on n . *This result requires a tight estimate of $B(x)$ such that $\text{wt}(x) = 1$, a technical contribution of our work.*

Substituting $\tau = \frac{\log(\frac{n}{4} \log \lambda)}{2 \log \lambda}$, we get that $2\lambda^\tau = (n \log \lambda)^{1/2}$. Henceforth, we will carry this upper bound as $(n \log \lambda)^{n/2} \cdot \lambda^{-n/2+1}$.

Remark 3. Determining the estimates $B(x)$ is equivalent to estimating the evaluations of Krawtchouk polynomials. We require concrete estimates, not asymptotics such as the one in [[Dom08](#)]. Elementary estimates suffice for our applications; tighter estimates, for example, those in [[Kra01](#), Section 3], do not yield any qualitative improvement of our results. We present upper bounds on such estimates in [Appendix E](#). \triangle

Remark 4 (A world without estimating the higher order spectral norm). Suppose we upper bound the second summation using prior techniques as used in [[BDir18](#), [BDir21](#), [MPSW21](#), [MNPW22](#), [FMM+24](#)]; they did not analyze the ℓ_n norm. They upper bound the second summand using the ℓ_∞ norm and the ℓ_2 norm of the Fourier coefficients. Using our tight estimate of $B(x) = \lambda^{-1}$, when $\text{wt}(x) = 1$, the old strategy will yield an upper bound of $\text{poly}(\log \lambda) \cdot \lambda^{-(n-2)-(1/2)+n/2}$, which will be $o(1)$ only for $n \geq 4$. Our upper bound is tighter; it is $o(1)$ for $n \geq 3$. See also [Appendix C.2](#). \triangle

First summand estimation. To summarize our derivation so far, for $n \geq 3$ parties, we have

$$2 \cdot \text{SD}(\vec{\text{wt}}(s), \vec{\text{wt}}(U_F)) \leq \underbrace{\sum_{\vec{w} \in \text{Typical}(n, \tau)} \sum_{\zeta \in \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right|}_{\text{first summand}} + \underbrace{\left((n \log \lambda)^{n/2} + 4n/\lambda \right) \cdot \lambda^{-n/2+1}}_{\text{small}}. \quad (9)$$

Note that to upper bound the first summand it suffices to show that the following quantity is small for every $\zeta \in \mathcal{S}_{\vec{\beta}}$:

$$\sum_{\vec{w} \in \{0, 1, \dots, \lambda\}^n} \prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right|.$$

Because, if the above quantity is small, then so is the summation restricted to $\vec{w} \in \text{Typical}(n, \tau)$. And, in turn, the expression in the first summand is small too, with an additional multiplicative factor of at most n (since $\text{card}(\mathcal{S}_{\vec{\beta}}) \leq n$).

To begin, for any $\zeta \in \mathcal{S}_{\vec{\beta}}$, rewrite

$$\sum_{\vec{w} \in \{0,1,\dots,\lambda\}^n} \prod_{i=1}^n |\widehat{\mathbb{1}}_{w_i}(\beta_i \zeta)| = \prod_{i=1}^n \left(\sum_{w_i \in \{0,1,\dots,\lambda\}} |\widehat{\mathbb{1}}_{w_i}(\beta_i \zeta)| \right) \quad (10)$$

Let $H(\zeta; \vec{\beta}) := \{i: \beta_i \zeta \neq \zeta^*\}$. Consider an index $i \notin H(\zeta; \vec{\beta})$. For such an i , we have $\beta_i \zeta = \zeta^*$ and, consequently, $|\widehat{\mathbb{1}}_{w_i}(\beta_i \zeta)| = |\widehat{\mathbb{1}}_{w_i}(\zeta^*)| = |\widehat{\mathbb{1}}_{w_i}(0)| = \binom{\lambda}{w_i} \cdot 2^{-\lambda}$, and therefore, $\sum_{w_i} |\widehat{\mathbb{1}}_{w_i}(\beta_i \zeta)| = \sum_{w_i} \binom{\lambda}{w_i} \cdot 2^{-\lambda} = 1$. As a result, the expression in Equation 10 is

$$\text{(Equation 10)} = \prod_{i \in H(\zeta; \vec{\beta})} \left(\sum_{w_i \in \{0,1,\dots,\lambda\}} |\widehat{\mathbb{1}}_{w_i}(\beta_i \zeta)| \right)$$

If all elements in $\vec{\beta}$ are identical, then, observe that $\mathcal{S}_{\vec{\beta}}$ is a singleton set and $H(\zeta; \vec{\beta}) = \emptyset$ – a lost case for us; the expression above is 1. On the other hand, if *its not the case that all elements in $\vec{\beta}$ are identical*, then $H(\zeta; \vec{\beta}) \neq \emptyset$ for every $\zeta \in \mathcal{S}_{\vec{\beta}}$. For any $i \in H(\zeta; \vec{\beta})$, it will suffice to prove that

$$\sum_{w_i \in \{0,1,\dots,\lambda\}} |\widehat{\mathbb{1}}_{w_i}(\beta_i \zeta)| = \mathcal{O}(\lambda)^{-1/4}.$$

Define $\zeta' := \beta_i \zeta$. Observe that $\zeta' \in F \setminus \{0, \zeta^*\}$. A standard application of the Parseval's identity yields the estimate:

$$|\widehat{\mathbb{1}}_{w_i}(\zeta')| \leq \sqrt{\binom{\lambda}{w_i} \cdot 2^{-\lambda} \cdot \binom{\lambda}{w'}^{-1}},$$

where $w' = \text{wt}(\zeta')$ and $w' \in \{1, 2, \dots, \lambda - 1\}$. Therefore, using the fact $\binom{\lambda}{w'} \geq \binom{\lambda}{1} = \lambda$, we have

$$|\widehat{\mathbb{1}}_{w_i}(\zeta')| \leq \sqrt{\binom{\lambda}{w_i} \cdot 2^{-\lambda} \cdot \lambda^{-1}},$$

Using this upper bound on the Fourier coefficient, we have

$$\begin{aligned} \sum_{w_i \in \{0,1,\dots,\lambda\}} |\widehat{\mathbb{1}}_{w_i}(\zeta')| &\leq 2^{-\lambda/2} \lambda^{-1/2} \sum_{w_i \in \{0,1,\dots,\lambda\}} \binom{\lambda}{w_i}^{1/2} \\ &< 2^{-\lambda/2} \lambda^{-1/2} \cdot \pi 2^{\lambda/2} \lambda^{1/4} = \pi \lambda^{-1/4} \end{aligned} \quad (\text{by Claim 5})$$

which is what we set out to prove.

Remark 5 (Perspective). Suppose $\vec{\beta}$ has $k \geq 2$ distinct elements $\beta_1, \beta_2, \dots, \beta_k$ and they occur n_1, n_2, \dots, n_k times in $\vec{\beta}$, respectively. Without loss of generality, assume that $1 \leq n_1 \leq n_2 \leq \dots \leq n_k$. Note that $n_1 + n_2 + \dots + n_k = n$ and $\mathcal{S}_{\vec{\beta}} = \{\zeta^* \beta_1^{-1}, \dots, \zeta^* \beta_k^{-1}\}$, a set of k elements. Furthermore, the set $H(\zeta^* \beta_i^{-1}; \vec{\beta})$ has cardinality $(n - n_i)$, for $i \in \{1, 2, \dots, k\}$. Therefore, our upper bound calculated above will be

$$\sum_{i=1}^k \left(\pi \cdot \lambda^{-1/4} \right)^{n - n_i}.$$

And, this upper bound is largest (a.k.a., the weakest) when $n_1 = \dots = n_{k-1} = 1$ and $n_k = (n-k+1)$. For this case, the upper bound becomes

$$\underbrace{\left(\pi \cdot \lambda^{-1/4}\right)^{(k-1)}}_{\text{dominant term}} + (k-1)\left(\pi \cdot \lambda^{-1/4}\right)^{n-1}$$

Overall, the worst upper bound happens where $\vec{\beta}$ has $k = 2$ distinct elements, one of them occurring once, and the other occurring $(n-1)$ times. Even in this worst case, the upper bound is $\mathcal{O}(\lambda^{-1/4})$. \triangle

Putting things together and concluding remarks. Substituting these estimates of the two summands, we get

$$2 \cdot \text{SD}(\vec{\text{wt}}(s), \vec{\text{wt}}(U_F)) \leq \pi \lambda^{-1/4} + \underbrace{\left((n \log \lambda)^{n/2} + 4n/\lambda \right)}_{\text{small}} \cdot \lambda^{-n/2+1}$$

The “small” quantity in the insecurity upper-bound is (roughly) $\lambda^{-n/2+1}$, which is $o(1)$, for $n \geq 3$. This wraps up the proof overview of [Theorem 1](#).

Which schemes are most secure? [Theorem 1](#) shows that $\text{Gen}(\text{Add}_n, \vec{\beta})$ is secure against Hamming weight leakage as long as the reconstruction multiplier $\vec{\beta}$ is not a constant vector; in other words, the only instance where the generalized additive secret sharing becomes insecure is when it is equivalent to the additive secret sharing. Does this mean all choices of $\vec{\beta}$ are equally secure, or can some choices offer even stronger security?

Which $\vec{\beta}$ provides $\text{Gen}(\text{Add}_n, \vec{\beta})$ the strongest security against Hamming weight leakage?

We identify the *score function* that quantifies the security of the $\text{Gen}(\text{Add}_n, \vec{\beta})$. This function captures how the choice of $\vec{\beta}$ influences the insecurity, as formalized in the following theorem.

Theorem 2. *For $\vec{\beta} \in (F^*)^n$, the $\text{Gen}(\text{Add}_n, \vec{\beta})$ secret sharing scheme is ε insecure against the Hamming weight leakage, where*

$$\varepsilon = \mathcal{O}(n \log \lambda)^{n/2} \cdot \exp\left(-\min_{\zeta \in \mathcal{S}_{\vec{\beta}}} \text{Score}(\zeta; \vec{\beta})\right) + \mathcal{O}(n \log \lambda)^{n/2} \cdot \lambda^{-n/2+1}.$$

In particular, if $\vec{\beta} \in (F^)^n$ satisfies*

$$\min_{\zeta \in \mathcal{S}_{\vec{\beta}}} \text{Score}(\zeta; \vec{\beta}) \geq \mathcal{O}(n \log \log \lambda) + cn \log \lambda \tag{11}$$

for $\frac{1}{2} - \frac{1}{n} > c > 0$, then $\text{Gen}(\text{Add}_n, \vec{\beta})$ is $\mathcal{O}(\lambda^{-cn})$ insecure against the Hamming weight leakage.

Choosing $\vec{\beta}$ more carefully, by maximizing the minimum score, can lead to a faster reduction in insecurity. We now present a different strategy to upper bound the first summand. Below is a brief overview of the proof of this theorem.

First summand estimation using an alternative approach. We upper bound the first summand in Equation 9 as follows:

$$\text{first summand} \leq \left(2\lambda^{1/2+\tau}\right)^n \cdot n \cdot \max_{\substack{\vec{w} \in \text{Typical}(n,\tau) \\ \zeta \in \mathcal{S}_{\vec{\beta}}}} \left(\prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right| \right)$$

The key to upper bounding the first summand is to understand how the monomial $\prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right|$ can become large. Recall that $\widehat{\mathbb{1}}_{w_i}(\zeta^*)$ has a large magnitude. If $\vec{\beta}$ is such that every $\zeta \beta_i$ in the monomial simultaneously becomes ζ^* then the monomial has large magnitude, and we cannot prove the security of the secret-sharing scheme. This is exactly what happens when $\vec{\beta} = \vec{1}$; or, more generally, when all elements in $\vec{\beta}$ are identical.

For $\zeta \in \mathcal{S}_{\vec{\beta}}$, we will assign a score $\text{Score}(\zeta; \vec{\beta})$ such that

$$\prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right| \leq \lambda^{-n/2} \cdot \exp\left(-\text{Score}(\zeta; \vec{\beta})\right)$$

Equation 3 presents the definition of our score function and Lemma 5 proves that it satisfies the equation above. This definition does not depend on w_i , only on Hamming weights of $\zeta \beta_i \in F^*$, where $i \in \{1, 2, \dots, n\}$. As a consequence, we get the following upper bound for the first summand, upon substituting our value of τ :

$$\text{first summand} \leq (n \log \lambda)^{n/2} n \cdot \exp\left(-\min_{\zeta \in \mathcal{S}_{\vec{\beta}}} \text{Score}(\zeta; \vec{\beta})\right).$$

If the minimum score for $\zeta \in \mathcal{S}_{\vec{\beta}}$ is $\geq c \cdot n \log \lambda$, for some $1/2 \geq c > 0$, then the scheme will have insecurity $\leq \lambda^{-cn}$. When the elements of $\vec{\beta}$ are all identical, then its minimum score is 0; in which case, the first summand is not small. Substituting this bound into Equation 9, we get our desired result.

Over arbitrary fields. We show that $\text{Gen}(\text{Add}_n, \vec{\beta})$ inherits its security from the security of Add_n , in the sense that $\text{Gen}(\text{Add}_n, \vec{\beta})$ is no less secure than the additive secret sharing Add_n over any finite field. We make this connection precise via a *diagonalization* argument that relates the insecurity bound of $\text{Gen}(\text{Add}_n, \vec{\beta})$ directly to the quantity that is closely related to the insecurity of Add_n .

Theorem 3 (Diagonalization). Let F_q be a finite field of order q . Let $\vec{\beta} = (\beta_1, \dots, \beta_n) \in (F_q^*)^n$ be multipliers and $\vec{\tau} = (\tau_1, \dots, \tau_n)$ be the leakage. Define

$$\mu_i := \sum_{\vec{\ell} \in \Omega^n} \sum_{\zeta \in F_q^*} \left| \widehat{\mathbb{1}}_{\tau_i^{-1}(\ell_i)}(\zeta) \right|^n.$$

Then,

$$\varepsilon(\text{Gen}(\text{Add}_n, \vec{\beta})) \leq \prod_{i=1}^n \mu_i^{1/n} \leq \frac{1}{n} \sum_{i=1}^n \mu_i \leq \max(\mu_1, \dots, \mu_n)$$

Remark 6. Combining Theorem 3 with the result of Faust et al. [FMM⁺24], we immediately obtain that $\text{Gen}(\text{Add}_n, \vec{\beta})$ over F_p , where $p = 2^\lambda - 1$ is a Mersenne prime, is $\mathcal{O}(\lambda^{-n/4})$ -insecure against Hamming weight leakage whenever $n \geq 5$, for all $\vec{\beta} \in (F_p^*)^n$ unless it is not a constant vector. \triangle

2 Preliminaries

Basic Notations. Let F_q be a finite field of order (cardinality) q and λ be an integer such that $2^{\lambda-1} < q \leq 2^\lambda$. Unless the order needs to be specified, we will simply write F to denote the finite field. Let $F^* := F \setminus \{0\}$ denote the multiplicative group of F . We use U_F to denote the uniform distribution over F , and we write $x \leftarrow U_F$ when sampling $x \in F$ uniformly at random.

Given a set S , we denote by $\mathbb{1}_S$ the indicator function for set S , which gives $\mathbb{1}_S(x) = 1$ if $x \in S$ and 0 otherwise. For any function $f: D \rightarrow R$ where D is the domain and R is the range, we write $f^{-1}(y) := \{x \in D: f(x) = y\}$ for the set of all preimages of $y \in R$. For instance, $x \in F$ has $\text{wt}(x) = w$ if and only if $\mathbb{1}_{\text{wt}^{-1}(w)}(x) = 1$. For brevity, we write $\mathbb{1}_w := \mathbb{1}_{\text{wt}^{-1}(w)}$, i.e.

$$\mathbb{1}_w(x) := \begin{cases} 1, & \text{if } \text{wt}(x) = w \\ 0, & \text{otherwise} \end{cases}$$

2.1 Secret Sharing (Masking) Schemes

Definition 1 (Additive Secret Sharing). *Let F be a finite field and n a positive integer. Given a secret $s \in F$, additive secret sharing Add_n shares s by sampling $(s_1, \dots, s_n) \in F^n$ uniformly at random conditioned on $s_1 + \dots + s_n = s$. More concretely, Add_n consists of the following two algorithms: given $s \in F$,*

- **Share**(s): *Sample $s_1, \dots, s_{n-1} \leftarrow U_F$, set $s_n = s - (s_1 + \dots + s_{n-1})$, then return (s_1, \dots, s_n) .*
- **Reconstruct**($\{s_1, \dots, s_n\}$): *Return $\tilde{s} = s_1 + \dots + s_n$.*

We denote by $\text{Add}_n(s)$ the resulting distribution of shares (s_1, \dots, s_n) .

Definition 2 (Generalized Additive Secret Sharing). *Let F be a finite field and n a positive integer. Given reconstruction multipliers $\vec{\beta} = (\beta_1, \dots, \beta_n) \in (F^*)^n$ and a secret $s \in F$, the generalized additive secret sharing $\text{Gen}(\text{Add}_n, \vec{\beta})$ shares s by sampling $(s_1, \dots, s_n) \in F^n$ uniformly at random conditioned on $\beta_1 s_1 + \dots + \beta_n s_n = s$. More concretely, $\text{Gen}(\text{Add}_n, \vec{\beta})$ consists of the following two algorithms:*

- **Share**(s): *Sample $s_1, \dots, s_{n-1} \leftarrow U_F$, set $s_n = s - \beta_n^{-1}(\beta_1 s_1 + \dots + \beta_{n-1} s_{n-1})$, then return (s_1, \dots, s_n) .*
- **Reconstruct**($\{s_1, \dots, s_n\}$): *Return $\tilde{s} = \beta_1 s_1 + \dots + \beta_n s_n$.*

We denote by $\text{Gen}(\text{Add}_n, \vec{\beta}; s)$ the resulting distribution of shares (s_1, \dots, s_n) .

Note that, by definition, $\text{Gen}(\text{Add}_n, \vec{\beta})$ sharing $s \in F$ with $\vec{\beta} = (c, \dots, c) \in (F^*)^n$ where $c \in F^*$ constant is equivalent to Add_n sharing $c^{-1}s$.

The term of *generalized* secret sharing was chosen in line with notational conventions in coding theory literature. Observe that, $\text{Gen}(\text{Add}_n, \vec{\beta})$ can be seen as Add_n generating the shares (s'_1, \dots, s'_n) such that $s'_1 + \dots + s'_n = s$, then postprocessing them into $(\beta_1^{-1}s'_1, \dots, \beta_n^{-1}s'_n)$. Interpreting the shares as codewords of a linear code [Mas95], this postprocessing corresponds to the notion of generalized linear codes; see, for example, [Hal03, Chapter 5].

Definition 3 (Shamir Secret Sharing). *Let F be a finite field and n and k be positive integers such that $k \leq n$. Denote by $F[X]_{<k}$ the set of polynomials over F of degree less than k [CDN15, Definition 11.6]. Given evaluation places $\vec{X} = (X_1, \dots, X_n) \in (F^*)^n$ and a secret $s \in F$, Shamir secret sharing $\text{Shamir}(n, k, \vec{X})$ shares s by:*

- **Share**(s): Sample a polynomial $P(X) \leftarrow F[X]_{<k}$ uniformly at random conditioned on $P(0) = s$. Evaluate $P(X)$ at each value in \vec{X} , i.e. $s_i = P(X_i)$ for $i = 1, \dots, n$. Return (s_1, \dots, s_n) .
- **Reconstruct**($\{s_{i_1}, \dots, s_{i_k}\}$): Interpolate the polynomial $\tilde{P}(X) \in F[X]_{<k}$ over $(X_{i_1}, s_{i_1}), \dots, (X_{i_k}, s_{i_k})$. Return $\tilde{s} := \tilde{P}(0)$.

We denote by $\text{Shamir}(n, k, \vec{X}; s)$ the resulting distribution of shares (s_1, \dots, s_n) .

Proposition 1. Shamir secret sharing with $k = n$ is an instance of generalized additive secret sharing. Therefore, proving security of $\text{Gen}(\text{Add}_n, \vec{\beta})$ automatically translates to the security of $\text{Shamir}(n, n, \vec{X})$.

Proof of Proposition 1. It suffices to show that, for all evaluation points $\vec{X} \in (F^*)^n$ and secret $s \in F$, there exists $\vec{\beta} \in (F^*)^n$ such that $\beta_1 s_1 + \dots + \beta_n s_n = s$ where $(s_1, \dots, s_n) \leftarrow \text{Shamir}(n, n, \vec{X}; s)$.

Given n pairs $(X_1, s_1), \dots, (X_n, s_n)$ from $\text{Shamir}(n, n, \vec{X})$, one can recover the polynomial $P(X)$ that was used to compute s_i 's via Lagrange interpolation:

$$P(X) := \sum_{i=1}^n \underbrace{\left(\prod_{j \in \{1, 2, \dots, n\} \setminus \{i\}} \frac{X - X_j}{X_i - X_j} \right)}_{=: L_i(X)} \cdot s_i = \sum_{i=1}^n L_i(X) \cdot s_i$$

and hence the secret $s = P(0) = \sum_{i=1}^n L_i(0) \cdot s_i$. Therefore, $\text{Shamir}(n, n, \vec{X})$ that shares s is $\text{Gen}(\text{Add}_n, \vec{\beta})$ sharing s , where

$$\beta_i = L_i(0) = \prod_{m \in \{1, 2, \dots, n\} \setminus \{i\}} \frac{X_m}{X_m - X_i}$$

for all $i \in \{1, 2, \dots, n\}$. □

2.2 Leakage-Resilient Secret Sharing

We quantify the variation between two distributions P and Q using the *statistical distance*.

Definition 4 (Statistical Distance). The statistical distance between two distributions P and Q over a finite space Ω is defined as

$$\text{SD}(P, Q) := \frac{1}{2} \sum_{x \in \Omega} \left| \Pr[P = x] - \Pr[Q = x] \right|.$$

Definition 5 (Local Leakage [BDIR18, BDIR21]). Let F be a finite field and n be a positive integer. Let \mathcal{S} be a secret sharing scheme with sharing algorithm **Share**. Let $\vec{\tau} = (\tau_1, \dots, \tau_n)$ be a collection of n functions whose domains are F . For any secret $s \in F$, the leakage distribution $\vec{\tau}(\text{Share}(s))$ is defined by the following experiment:

- (1) Sample shares $\vec{s} = (s_1, \dots, s_n)$ from $\text{Share}(s)$.
- (2) Output $(\tau_1(s_1), \dots, \tau_n(s_n))$.

We refer to $\vec{\tau}(\text{Share}(s))$ as the joint leakage distribution induced by all shares of $s \in F$, and for brevity we shall also write it as $\vec{\tau}(s)$ instead.

Definition 6 (ε -insecurity). A secret sharing scheme \mathcal{S} is said to be ε -insecure against a local leakage $\vec{\tau}$ if for all secret $s \in F$,

$$\varepsilon(\mathcal{S}; \vec{\tau}) := 2 \cdot \text{SD}(\vec{\tau}(\text{Share}(s)), \vec{\tau}(\text{Share}(U_F))) \leq \varepsilon$$

Equivalently, for all pairs of secrets $(s^{(0)}, s^{(1)})$,

$$2 \cdot \text{SD}(\vec{\tau}(\text{Share}(s^{(0)})), \vec{\tau}(\text{Share}(s^{(1)}))) \leq \varepsilon$$

Roughly speaking, if \mathcal{S} has small insecurity against a leakage attack, then the joint distribution of leakage of shares is statistically independent of the secret.

In this paper, we primarily focus on the setting where all leakage functions are the Hamming weight, i.e. $\tau_1 = \dots = \tau_n = \text{wt}$. And in this case, $\vec{\tau}(\vec{s}) = \vec{\text{wt}}(\vec{s})$ corresponds to the joint Hamming weight leakage of the shares of s .

Definition 7 (Typical Weights). We call $w \in \{0, 1, \dots, \lambda\}$ to be typical if $|w - \lambda/2| \leq \lambda^{1/2+\tau}$ for some $\tau > 0$. And we say a vector $\vec{w} = (w_1, w_2, \dots, w_n) \in \{0, 1, \dots, \lambda\}^n$ is in a typical set $\text{Typical}(n, \tau)$ if every w_i is typical:

$$\text{Typical}(n, \tau) := \left\{ \vec{w} : |w_i - \lambda/2| \leq \lambda^{1/2+\tau} \quad \forall i \in \{1, 2, \dots, n\} \right\}.$$

2.3 Fourier Analysis over Finite Field

We will use Fourier analysis over the additive group $(F_{p^d}, +)$ for a prime $p \geq 2$ and degree of extension $d \in \{1, 2, \dots\}$. In this subsection, $F := F_{p^d}$.

Let $\omega_p := \exp(2\pi i/p)$. Define the Fourier transformation of $f: F \rightarrow F$ over F , denoted $\hat{f}: F \rightarrow \mathbb{C}$, as follows:

$$\hat{f}(\alpha) := \frac{1}{q} \sum_{x \in F} f(x) \cdot \omega_p^{-\text{Tr}_{F/F_p}(\alpha x)} \quad \forall \alpha \in F$$

We call $\chi_\alpha(x) = \omega_p^{\text{Tr}_{F/F_p}(\alpha x)}$ the character and $\hat{f}(\alpha)$ the Fourier coefficient of f at α . Here, $\text{Tr}_{F/F_p}(\alpha x)$ is a function defined as follows:

Definition 8 (Field Trace). The trace of an extension field $F = F_{p^d}$ over the base field F_p , denoted by $\text{Tr}_{F/F_p}: F \rightarrow F_p$, is defined as $\text{Tr}_{F/F_p}(y) := \sum_{i=0}^{d-1} y^{p^i}$.

For example, for $F = F_{2^d}$, $\omega := \omega_2 = \exp(\pi i) = -1$ and the Fourier coefficients and characters are

$$\hat{f}(\alpha) := \frac{1}{q} \sum_{x \in F_{2^d}} f(x) \cdot (-1)^{\langle x, \alpha \rangle}$$

where $\langle x, \alpha \rangle \in F_2$ denotes the inner product of x and α viewed as vectors. This is possible because the extension field F_{p^d} is isomorphic to $(F_p)^d$, the vector space of dimension d with base field F_p .

Fact 1 (Parseval's Identity). $\frac{1}{\text{card}(F)} \sum_{x \in F} |f(x)|^2 = \sum_{\alpha \in F} |\hat{f}(\alpha)|^2$.

Fact 2 (Character Sum). For all $\alpha \in F$,

$$\sum_{x \in F} \chi_\alpha(x) = \begin{cases} \text{card}(F) & \text{if } \alpha = 0 \\ 0 & \text{otherwise} \end{cases}$$

Note also that the modulus of character $|\chi_\alpha(x)| = 1$ for any α and x in F .

3 Insecurity Analysis

This section presents the full proof of main results ([Theorem 1](#) and [Theorem 2](#)) outlined in the technical overview ([Section 1.3](#)).

3.1 Worst-case Analysis ([Theorem 1](#))

Theorem 1. For $\vec{\beta} \in (F^*)^n \setminus \{(c, \dots, c) : c \in F^*\}$ and $n \in \{3, 4, \dots\}$, we have

$$\varepsilon\left(\text{Gen}(\text{Add}_n, \vec{\beta}) ; \vec{\text{wt}}\right) = \mathcal{O}\left(\lambda^{-1/4}\right).$$

Proof of [Theorem 1](#). By definition, it suffices to prove that the following quantity is upper bounded by ε .

$$\begin{aligned} & 2 \cdot \text{SD}(\vec{\text{wt}}(s), \vec{\text{wt}}(U_F)) \\ &= \sum_{\vec{w} \in \{0, 1, \dots, \lambda\}^n} \left| \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; s)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] - \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; U_F)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] \right| \end{aligned} \quad (12)$$

Recall from [Definition 7](#) that a weight $w \in \{0, 1, \dots, \lambda\}$ is said to be typical if it is only up to $\lambda^{1/2+\tau}$ away from the average $\lambda/2$ for some $\tau > 0$. Let us write $\vec{w} \notin \text{Typical}(n, \tau)$ to denote that $\vec{w} \in \{0, 1, \dots, \lambda\}^n \setminus \text{Typical}(n, \tau)$. Chernoff bound ([Lemma 1](#)) implies that the probability mass on atypical weights contribute only negligibly to the statistical distance.

$$\begin{aligned} & \sum_{\vec{w} \notin \text{Typical}(n, \tau)} \left| \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; s)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] - \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; U_F)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] \right| \\ & \leq \sum_{\vec{w} \notin \text{Typical}(n, \tau)} \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; s)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] \\ & \quad + \sum_{\vec{w} \notin \text{Typical}(n, \tau)} \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; U_F)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] \quad (\text{by triangle inequality}) \\ & \leq 2 \cdot \max_{s \in F} \left(\sum_{\vec{w} \notin \text{Typical}(n, \tau)} \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; s)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] \right) \\ & = 2 \cdot \sum_{\vec{w} \notin \text{Typical}(n, \tau)} \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; \tilde{s})} [\vec{\text{wt}}(\vec{s}) = \vec{w}] \quad (\text{let } \tilde{s} \text{ be the arg max}) \\ & = 2 \cdot \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; \tilde{s})} [\vec{\text{wt}}(\vec{s}) \notin \text{Typical}(n, \tau)] \\ & = 2 \cdot \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; \tilde{s})} [\text{wt}(s_1), \dots, \text{or } \text{wt}(s_n) \text{ is not typical}] \quad (\text{definition of Typical}) \\ & \leq 2 \cdot \sum_{j=1}^n \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; \tilde{s})} [\text{wt}(s_j) \text{ is not typical}] \quad (\text{by union bound}) \\ & \leq 2n \cdot 2 \exp(-2\lambda t^2) \quad (\text{by Chernoff bound ([Lemma 1](#)))} \end{aligned}$$

Hence, the expression for the statistical distance in [Equation 12](#) becomes a summation over a

typical set $\text{Typical}(n, \tau)$ plus a negligible quantity.

(Equation 12)

$$\begin{aligned}
&= \sum_{\vec{w} \in \{0,1,\dots,\lambda\}^n} \left| \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; s)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] - \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; U_F)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] \right| \\
&= \sum_{\vec{w} \in \text{Typical}(n, \tau)} \left| \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; s)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] - \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; U_F)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] \right| \\
&\quad \pm \sum_{\vec{w} \notin \text{Typical}(n, \tau)} \left| \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; s)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] - \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; U_F)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] \right| \\
&= \sum_{\vec{w} \in \text{Typical}(n, \tau)} \left| \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; s)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] - \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; U_F)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] \right| \\
&\quad \pm 2 \cdot 2n \cdot \exp(-2\lambda^2\tau)
\end{aligned} \tag{13}$$

If $\vec{s} \in F^n$ is a share of secret 0, i.e. $\vec{s} \in C := \text{Gen}(\text{Add}_n, \vec{\beta}; 0)$, then $\beta_1 s_1 + \dots + \beta_n s_n = 0$, and for any $\zeta \in F$, we have $\zeta \beta_1 s_1 + \dots + \zeta \beta_n s_n = \zeta \cdot 0 = 0$. From this, we can deduce $C^\perp = \{\zeta \vec{\beta} \mid \zeta \in F\}$, then by Poisson summation formula (Lemma 2), for any $\vec{v} \in \text{Gen}(\text{Add}_n, \vec{\beta}; 1)$ we have

$$\begin{aligned}
\Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; s)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] &= \sum_{\vec{z} \in C^\perp} \left(\prod_{i=1}^n \hat{\mathbb{1}}_{w_i}(z_i) \right) \cdot \chi_1(s \cdot \langle \vec{z}, \vec{v} \rangle) \\
&= \sum_{\zeta \in F} \left(\prod_{i=1}^n \hat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right) \cdot \chi_1(s \cdot \zeta \cdot \langle \vec{\beta}, \vec{v} \rangle) \\
&= \sum_{\zeta \in F} \left(\prod_{i=1}^n \hat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right) \cdot \chi_1(s \cdot \zeta) \quad (\text{because } \vec{v} \in \text{Gen}(\text{Add}_n, \vec{\beta}; 1)) \\
&= \sum_{\zeta \in F} \left(\prod_{i=1}^n \hat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right) \cdot \chi_\zeta(s)
\end{aligned} \tag{14}$$

Moreover,

$$\begin{aligned}
\Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; U_F)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] &= \frac{1}{\text{card}(F)} \sum_{s \in F} \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; s)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] \\
&= \frac{1}{\text{card}(F)} \sum_{s \in F} \sum_{\zeta \in F} \left(\prod_{i=1}^n \hat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right) \cdot \chi_\zeta(s) \quad (\text{by Equation 14}) \\
&= \frac{1}{\text{card}(F)} \sum_{\zeta \in F} \left(\prod_{i=1}^n \hat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right) \cdot \sum_{s \in F} \chi_\zeta(s) \\
&= \frac{1}{\text{card}(F)} \sum_{\zeta \in \{0\}} \left(\prod_{i=1}^n \hat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right) \cdot \text{card}(F) \quad (\text{by Fact 2}) \\
&= \sum_{\zeta \in \{0\}} \left(\prod_{i=1}^n \hat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right)
\end{aligned} \tag{15}$$

Plugging these into the summand in [Equation 13](#), we get

$$\begin{aligned}
& \left| \sum_{\vec{w} \in \text{Typical}(n, \tau)} \left| \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; s)} [\text{wt}(\vec{s}) = \vec{w}] - \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; U_F)} [\text{wt}(\vec{s}) = \vec{w}] \right| \right| \\
&= \sum_{\vec{w} \in \text{Typical}(n, \tau)} \left| \sum_{\zeta \in F} \left(\prod_{i=1}^n \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right) \cdot \chi_\zeta(s) - \sum_{\zeta \in \{0\}} \left(\prod_{i=1}^n \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right) \right| \quad (\text{by Equation 14 and 15}) \\
&= \sum_{\vec{w} \in \text{Typical}(n, \tau)} \left| \sum_{\zeta \in F^*} \left(\prod_{i=1}^n \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right) \cdot \chi_\zeta(s) \right| \quad (\text{since } \chi_0(s) = 1 \text{ for all } s \in F) \\
&\leq \sum_{\vec{w} \in \text{Typical}(n, \tau)} \sum_{\zeta \in F^*} \prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right| \quad (\text{by triangle inequality}) \quad (16)
\end{aligned}$$

Let $\zeta^* \in F$ be the element such that $\text{wt}(\zeta^*) = \lambda$ and consider the set

$$\mathcal{S}_{\vec{\beta}} := \left\{ \zeta^* \cdot \beta_1^{-1}, \zeta^* \cdot \beta_2^{-1}, \dots, \zeta^* \cdot \beta_n^{-1} \right\}.$$

Then for any $\zeta \in \mathcal{S}_{\vec{\beta}}$, at least one of β_i 's should give $\beta_i \zeta = \zeta^*$. Consider the following separation of the summation.

$$(\text{Equation 16}) = \sum_{\vec{w} \in \text{Typical}(n, \tau)} \sum_{\zeta \in \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right| + \sum_{\vec{w} \in \text{Typical}(n, \tau)} \sum_{\zeta \in F^* \setminus \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right| \quad (17)$$

Let us upper bound the second summand (the summation over $F^* \setminus \mathcal{S}_{\vec{\beta}}$). As stated in [Lemma 4](#), we denote $B(\zeta)$ to be a function that upper bounds $|\widehat{\mathbb{1}}_w(\zeta)|$. Then,

$$\begin{aligned}
& \sum_{\vec{w} \in \text{Typical}(n, \tau)} \sum_{\zeta \in F^* \setminus \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right| \\
&\leq \sum_{\vec{w} \in \text{Typical}(n, \tau)} \sum_{\zeta \in F^* \setminus \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n B(\beta_i \zeta) \\
&= \left(2\lambda^{1/2+\tau} \right)^n \sum_{\zeta \in F^* \setminus \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n B(\beta_i \zeta) \quad (\text{since } \text{card}(\text{Typical}(n, \tau)) = (2\lambda^{1/2+\tau})^n) \\
&\leq \left(2\lambda^{1/2+\tau} \right)^n \sum_{\zeta \in F \setminus \{0, \zeta^*\}} B(\zeta)^n \quad (\text{by Hölder's inequality (Lemma 3)}) \\
&\leq 2^{n+1} \cdot 5^{\frac{5n}{2}-4} \cdot \lambda^{n\tau} \cdot \lambda^{-\frac{n}{2}+1} \quad (\text{by Lemma 4}) \quad (18)
\end{aligned}$$

The first summand (the summation over $\mathcal{S}_{\vec{\beta}}$) can be rewritten as follows

$$\begin{aligned}
& \sum_{\vec{w} \in \text{Typical}(n, \tau)} \sum_{\zeta \in \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right| \\
&\leq \sum_{\vec{w} \in \{0, 1, \dots, \lambda\}^n} \sum_{\zeta \in \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right| \quad (\text{because } \text{Typical}(n, \tau) \subseteq \{0, 1, \dots, \lambda\}^n \text{ and } |\widehat{\mathbb{1}}_{w_i}(\beta_i \zeta)| \geq 0)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\zeta \in \mathcal{S}_{\vec{\beta}}} \sum_{\vec{w} \in \{0,1,\dots,\lambda\}^n} \prod_{i=1}^n |\widehat{\mathbb{1}}_{w_i}(\beta_i \zeta)| \\
&= \sum_{\zeta \in \mathcal{S}_{\vec{\beta}}} \prod_{i \in \{1,2,\dots,n\}} \sum_{w_i=0}^{\lambda} |\widehat{\mathbb{1}}_{w_i}(\beta_i \zeta)| \\
&= \sum_{\zeta \in \mathcal{S}_{\vec{\beta}}} \left(\prod_{i \in H(\zeta; \vec{\beta})} \sum_{w_i=0}^{\lambda} |\widehat{\mathbb{1}}_{w_i}(\beta_i \zeta)| \right) \cdot \left(\prod_{i \notin H(\zeta; \vec{\beta})} \sum_{w_i=0}^{\lambda} |\widehat{\mathbb{1}}_{w_i}(\beta_i \zeta)| \right) \tag{19}
\end{aligned}$$

where $H(\zeta; \vec{\beta}) := \{i: \beta_i \zeta \neq \zeta^*\}$. Next, we separate bound the ℓ_1 -norms on the right-hand side for those indices $i \in H(\zeta; \vec{\beta})$ and $i \notin H(\zeta; \vec{\beta})$.

Case 1. Consider $i \in H(\zeta; \vec{\beta})$. This is the non-trivial case, we want a non-trivial upper bound.

$$\begin{aligned}
\sum_{w_i=0}^{\lambda} |\widehat{\mathbb{1}}_{w_i}(\beta_i \zeta)| &\leq \sum_{w_i=0}^{\lambda} \binom{\lambda}{w_i}^{1/2} \cdot 2^{-\lambda/2} \cdot \left(\binom{\lambda}{\text{wt}(\beta_i \zeta)} \right)^{-1/2} && \text{(by Claim 2)} \\
&= \left(\binom{\lambda}{\text{wt}(\beta_i \zeta)} \right)^{-1/2} \sum_{w_i=0}^{\lambda} \binom{\lambda}{w_i}^{1/2} \cdot 2^{-\lambda/2} \\
&< \left(\binom{\lambda}{\text{wt}(\beta_i \zeta)} \right)^{-1/2} \cdot \pi \cdot \lambda^{1/4} && \text{(by Claim 5)}
\end{aligned}$$

Case 2. Consider $i \notin H(\zeta; \vec{\beta})$. In this case, we have the trivial estimate.

$$\sum_{w_i=0}^{\lambda} |\widehat{\mathbb{1}}_{w_i}(\beta_i \zeta)| = \sum_{w_i=0}^{\lambda} |\widehat{\mathbb{1}}_{w_i}(\zeta^*)| = \sum_{w_i=0}^{\lambda} \widehat{\mathbb{1}}_{w_i}(0) = \sum_{w_i=0}^{\lambda} \binom{\lambda}{w_i} \cdot 2^{-\lambda} = 1 \quad \text{(follows from Claim 2)}$$

Substituting these values into Equation 19 and continuing the upper bound, we have:

$$\text{(Equation 19)} < \sum_{\zeta \in \mathcal{S}_{\vec{\beta}}} \prod_{i \in H(\zeta; \vec{\beta})} \left(\binom{\lambda}{\text{wt}(\beta_i \zeta)} \right)^{-1/2} \cdot \pi \cdot \lambda^{1/4} \tag{20}$$

Note that, for any $i \in H(\zeta; \vec{\beta})$, $\text{wt}(\beta_i \zeta) \in \{1, 2, \dots, \lambda-1\}$. Therefore, using the fact that $\binom{\lambda}{\text{wt}(\beta_i \zeta)} \geq \binom{\lambda}{1} = \lambda$ for any $i \in H(\zeta; \vec{\beta})$, we obtain the following bound for the above quantity:

$$\text{(Equation 20)} \leq \sum_{\zeta \in \mathcal{S}_{\vec{\beta}}} \lambda^{-\frac{\text{card}(H(\zeta; \vec{\beta}))}{2}} \cdot \pi^{\text{card}(H(\zeta; \vec{\beta}))} \cdot \lambda^{\frac{\text{card}(H(\zeta; \vec{\beta}))}{4}} \leq (\tilde{h} + 1) \cdot \left(\frac{\pi^4}{\lambda} \right)^{\frac{\tilde{h}}{4}} \tag{21}$$

because $\text{card}(\mathcal{S}_{\vec{\beta}}) \leq \tilde{h} + 1$, where $\tilde{h} = \min_{\zeta \in \mathcal{S}_{\vec{\beta}}} \left(\text{card}(H(\zeta; \vec{\beta})) \right)$. ^[2]

^[2]If $\zeta \in \mathcal{S}_{\vec{\beta}}$, then there should exist $j \in \{1, 2, \dots, n\}$ such that $\zeta = \beta_j^{-1} \zeta^*$, and so $i \in H(\zeta; \vec{\beta})$ iff $\beta_i \neq \beta_j$. Hence, for any $\zeta \in \mathcal{S}_{\vec{\beta}}$, $\text{card}(H(\zeta; \vec{\beta})) \geq \text{card}(\mathcal{S}_{\vec{\beta}}) - 1$, making $\tilde{h} := \min_{\zeta \in \mathcal{S}_{\vec{\beta}}} \text{card}(H(\zeta; \vec{\beta})) \geq \text{card}(\mathcal{S}_{\vec{\beta}}) - 1$ as well.

Therefore, for a sufficiently large λ , the above sum is small as long as we have $\text{card}(H(\zeta; \vec{\beta})) \geq 1$; that is, unless $\beta_1 = \beta_2 = \dots = \beta_n$ (i.e., unless the secret sharing is the additive secret sharing), the above sum would be at most $\lambda^{-1/4}$. More precisely:

$$\begin{aligned}
& 2 \cdot \text{SD}(\vec{\text{wt}}(s), \vec{\text{wt}}(U_F)) \\
& \leq \sum_{\vec{w} \in \text{Typical}(n, \tau)} \sum_{\zeta \in \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right| + \sum_{\vec{w} \in \text{Typical}(n, \tau)} \sum_{\zeta \in F^* \setminus \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right| \\
& \quad + 2 \cdot 2n \cdot \exp(-2\lambda^{2\tau}) \quad (\text{by Equation 13 and 17}) \\
& \leq (\tilde{h} + 1) \cdot \left(\frac{\pi^4}{\lambda} \right)^{\frac{\tilde{h}}{4}} + 2^{n+1} \cdot 5^{\frac{5n}{2}-4} \cdot \lambda^{n\tau} \cdot \lambda^{-\frac{n}{2}+1} + 2 \cdot 2n \cdot \exp(-2\lambda^{2\tau}) \\
& \quad (\text{by Equation 18 and 21; also recall that } \tilde{h} := \min_{\zeta \in \mathcal{S}_{\vec{\beta}}} \text{card}(H(\zeta; \vec{\beta}))) \\
& \leq 2\pi \cdot \frac{c}{\lambda^{1/4}} + 2^{n+1} \cdot 5^{\frac{5n}{2}-4} \cdot \lambda^{n\tau} \cdot \lambda^{-\frac{n}{2}+1} + 2 \cdot 2n \cdot \exp(-2\lambda^{2\tau}) \\
& \quad (\text{because } \vec{\beta} \text{ cannot be of the form } (b, b, \dots, b) \in (F^*)^n)
\end{aligned}$$

which is $\mathcal{O}(\lambda^{-1/4})$, for sufficiently large λ . \square

3.2 Security Characterization (Theorem 2)

Theorem 2. For $\vec{\beta} \in (F^*)^n$, the $\text{Gen}(\text{Add}_n, \vec{\beta})$ secret sharing scheme is ε insecure against the Hamming weight leakage, where

$$\varepsilon = \mathcal{O}(n \log \lambda)^{n/2} \cdot \exp\left(-\min_{\zeta \in \mathcal{S}_{\vec{\beta}}} \text{Score}(\zeta; \vec{\beta})\right) + \mathcal{O}(n \log \lambda)^{n/2} \cdot \lambda^{-n/2+1}.$$

In particular, if $\vec{\beta} \in (F^*)^n$ satisfies

$$\min_{\zeta \in \mathcal{S}_{\vec{\beta}}} \text{Score}(\zeta; \vec{\beta}) \geq \mathcal{O}(n \log \log \lambda) + cn \log \lambda \quad (11)$$

for $\frac{1}{2} - \frac{1}{n} > c > 0$, then $\text{Gen}(\text{Add}_n, \vec{\beta})$ is $\mathcal{O}(\lambda^{-cn})$ insecure against the Hamming weight leakage.

Proof of Theorem 2. The proof goes in the same way as for Theorem 1, with the estimation of the second summand unchanged. The only difference lies in the first summand, which naturally gives rise to the score function quantifying the security of $\text{Gen}(\text{Add}_n, \vec{\beta})$.

The first summand (the summation over $\mathcal{S}_{\vec{\beta}}$) can be rewritten as follows

$$\begin{aligned}
\sum_{\vec{w} \in \text{Typical}(n, \tau)} \sum_{\zeta \in \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right| & \leq 2^n \cdot \lambda^{(\frac{1}{2}+\tau) \cdot n} \cdot n \cdot \max_{\substack{\vec{w} \in \text{Typical}(n, \tau) \\ \zeta \in \mathcal{S}_{\vec{\beta}}}} \prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right| \\
& \leq 2^n \cdot \lambda^{(\frac{1}{2}+\tau) \cdot n} \cdot n \cdot \lambda^{-\frac{n}{2}} \cdot \exp\left(-\min_{\zeta \in \mathcal{S}_{\vec{\beta}}} \text{Score}(\zeta; \vec{\beta})\right) \quad (\text{by Lemma 5}) \\
& = (2\lambda^\tau)^n \cdot n \cdot \exp\left(-\min_{\zeta \in \mathcal{S}_{\vec{\beta}}} \text{Score}(\zeta; \vec{\beta})\right) \quad (22)
\end{aligned}$$

because the maximum of a sequence is at least as large as its average. Hence, upon choosing $\tau = \frac{\log(\frac{n}{4} \log \lambda)}{2 \log \lambda}$ (so that $2\lambda^\tau = (n \log \lambda)^{1/2}$):

(Equation 17)

$$\begin{aligned} &\leq (2\lambda^\tau)^n \cdot n \cdot \exp\left(-\min_{\zeta \in \mathcal{S}_{\vec{\beta}}} \text{Score}(\zeta; \vec{\beta})\right) + 2 \cdot 5^{\frac{5n}{2}-4} \cdot (2\lambda^\tau)^n \cdot \lambda^{-\frac{n}{2}+1} \\ &\hspace{15em} \text{(by Equation 18 and 22)} \\ &\leq (n \log \lambda)^{n/2} \cdot n \cdot \exp\left(-\min_{\zeta \in \mathcal{S}_{\vec{\beta}}} \text{Score}(\zeta; \vec{\beta})\right) + 2 \cdot 5^{\frac{5n}{2}-4} \cdot (n \log \lambda)^{n/2} \cdot \lambda^{-\frac{n}{2}+1} \end{aligned} \quad (23)$$

In summary, we have the following chain of expressions:

$$\begin{aligned} &2 \cdot \text{SD}(\vec{\text{wt}}(s), \vec{\text{wt}}(U_F)) \\ &= \sum_{\vec{w} \in \{0,1,\dots,\lambda\}^n} \left| \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; s)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] - \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; U_F)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] \right| \quad \text{(from Equation 12)} \\ &= \sum_{\vec{w} \in \text{Typical}(n, \tau)} \left| \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; s)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] - \Pr_{\vec{s} \leftarrow \text{Gen}(\text{Add}_n, \vec{\beta}; U_F)} [\vec{\text{wt}}(\vec{s}) = \vec{w}] \right| \\ &\quad \pm 2 \cdot 2n \cdot \exp(-2\lambda^{2\tau}) \quad \text{(from Equation 13)} \\ &\leq \sum_{\vec{w} \in \text{Typical}(n, \tau)} \sum_{\zeta \in F^*} \prod_{i=1}^n \left| \hat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right| + 2 \cdot 2n \cdot \exp(-2\lambda^{2\tau}) \quad \text{(from Equation 16)} \\ &= \sum_{\vec{w} \in \text{Typical}(n, \tau)} \sum_{\zeta \in \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n \left| \hat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right| + \sum_{\vec{w} \in \text{Typical}(n, \tau)} \sum_{\zeta \in F^* \setminus \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n \left| \hat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right| + 2 \cdot 2n \cdot \exp(-2\lambda^{2\tau}) \\ &\hspace{15em} \text{(from Equation 17)} \\ &\leq (n \log \lambda)^{\frac{n}{2}} \cdot n \cdot \exp\left(-\min_{\zeta \in \mathcal{S}_{\vec{\beta}}} \text{Score}(\zeta; \vec{\beta})\right) + 2 \cdot 5^{\frac{5n}{2}-4} \cdot (n \log \lambda)^{\frac{n}{2}} \cdot \lambda^{-\frac{n}{2}+1} + 2 \cdot 2n \cdot \lambda^{-\frac{n}{2}} \\ &\hspace{15em} \text{(from Equation 23 and } \tau = \frac{\log(\frac{n}{4} \log \lambda)}{2 \log \lambda} \text{)} \\ &\leq (n \log \lambda)^{\frac{n}{2}} \cdot n \cdot \exp\left(-\min_{\zeta \in \mathcal{S}_{\vec{\beta}}} \text{Score}(\zeta; \vec{\beta})\right) + \left(5^{\frac{5n}{2}-3} \cdot (n \log \lambda)^{\frac{n}{2}} + \frac{4n}{\lambda}\right) \lambda^{-\frac{n}{2}+1} \\ &\leq \mathcal{O}(n \log \lambda)^{\frac{n}{2}} \cdot \exp\left(-\min_{\zeta \in \mathcal{S}_{\vec{\beta}}} \text{Score}(\zeta; \vec{\beta})\right) + \mathcal{O}(n \log \lambda)^{\frac{n}{2}} \cdot \lambda^{-\frac{n}{2}+1} \quad \text{(for sufficiently large } \lambda \text{)} \end{aligned}$$

Now, suppose that $\vec{\beta}$ gives

$$\min_{\zeta \in \mathcal{S}_{\vec{\beta}}} \text{Score}(\zeta; \vec{\beta}) \geq \left(\frac{n}{2} + 1\right) \log(n \log \lambda) + cn \log \lambda \quad (24)$$

for some constant c . Then, from the above expression,

$$\begin{aligned} &2 \cdot \text{SD}(\vec{\text{wt}}(s), \vec{\text{wt}}(U_F)) \\ &\leq (n \log \lambda)^{\frac{n}{2}} \cdot n \cdot \exp\left(-\min_{\zeta \in \mathcal{S}_{\vec{\beta}}} \text{Score}(\zeta; \vec{\beta})\right) + \left(5^{\frac{5n}{2}-3} \cdot (n \log \lambda)^{\frac{n}{2}} + \frac{4n}{\lambda}\right) \cdot \lambda^{-\frac{n}{2}+1} \end{aligned}$$

$$\begin{aligned}
&\leq (n \log \lambda)^{\frac{n}{2}} \cdot n \cdot (n \log \lambda)^{-\frac{n}{2}-1} \cdot \lambda^{-cn} + \left(5^{\frac{5n}{2}-3} \cdot (n \log \lambda)^{\frac{n}{2}} + \frac{4n}{\lambda}\right) \cdot \lambda^{-\frac{n}{2}+1} \\
&\leq \lambda^{-cn} + \mathcal{O}\left(\lambda^{-c'n}\right) = \mathcal{O}\left(\lambda^{-cn}\right) \quad \left(\frac{\log \lambda}{\lambda} \leq \lambda^{-c'} \text{ holds for all } c' \in (0, 1/2)\right)
\end{aligned}$$

and this concludes the proof of [Theorem 2](#). \square

[Theorem 2](#) provides a method for choosing reconstruction multipliers $\vec{\beta} \in (F^*)^n$ such that $\text{Gen}(\text{Add}_n, \vec{\beta})$ is guaranteed to be secure.

Corollary 1. *Let $n \geq 3$ and $\vec{\beta} = (\beta_1, \dots, \beta_n) \in (F^*)^n$ where there exists β_i and β_j such that*

$$4cn \leq \text{wt}(\beta_i \zeta^* \beta_j^{-1}) \leq \lambda - 4cn \quad \text{and} \quad 4cn \leq \text{wt}(\beta_j \zeta^* \beta_i^{-1}) \leq \lambda - 4cn \quad (25)$$

for $c \in (0, 1/2]$. Then, $\text{Gen}(\text{Add}_n, \vec{\beta})$ is λ^{-cn} -insecure against Hamming weight leakage.

Proof of Corollary 1. By definition ([Equation 2](#)), as long as $\lambda \geq (4cn)^2$,

$$\sigma(\text{wt}(\beta_i \zeta^* \beta_j^{-1})) \geq cn \log \lambda \quad \text{and} \quad \sigma(\text{wt}(\beta_j \zeta^* \beta_i^{-1})) \geq cn \log \lambda$$

Then we get

$$\min_{\zeta \in \mathcal{S}_{\vec{\beta}}} \text{Score}(\zeta; \vec{\beta}) \geq 2cn \log \lambda$$

and it satisfies [Equation 24](#) in the proof of [Theorem 2](#). Therefore, $\text{Gen}(\text{Add}_n, \vec{\beta})$ is at most λ^{-cn} -insecure by [Theorem 2](#). \square

Remark 7 (Security of Shamir Secret Sharing). [Corollary 1](#) also implies the security of $\text{Shamir}(n, n, \vec{X})$ when the evaluation places $\vec{X} = (X_1, \dots, X_n)$ are chosen uniformly at random.

Observe that, if β_i and β_j are chosen independently at random from F , the constraints in [Equation 25](#) are satisfied with probability exponentially close to 1 by Chernoff bound.

Recall from the proof of [Proposition 1](#) that $\text{Shamir}(n, n, \vec{X})$ can be reduced to $\text{Gen}(\text{Add}_n, \vec{\beta})$ such that $\beta_i = \prod_{m \in \{1, 2, \dots, n\} \setminus \{i\}} \frac{X_m}{X_m - X_i}$. Hwang et al. [[HMNY25](#)] showed that if X_1, \dots, X_n are chosen independently and uniformly at random, then β_1, \dots, β_n are pairwise uniform; that is, any β_i and β_j are (exponentially close to being) independent and uniform over F . Therefore, [Equation 25](#) is satisfied with very high probability, from which we can conclude that $\text{Shamir}(n, n, \vec{X})$ is secure against Hamming weight leakage for almost all \vec{X} .

In addition, for security certification, note that the mapping $\vec{X} \mapsto \vec{\beta}$ is efficiently computable; hence, given \vec{X} , one can simply compute $\vec{\beta}$ from \vec{X} , evaluate the score of $\vec{\beta}$, and from there one can verify the security of \vec{X} . \triangle

3.3 Towards Generalization

The previous two results focused on enhancing the security of additive secret sharing over the binary extension field $F = F_{2^\lambda}$ by introducing additional twists called multiplicities $\vec{\beta} = (\beta_1, \dots, \beta_n)$. It is an appealing ansatz to hope that such a relationship carries over to arbitrary fields. In this section, we prove that if Add_n is secure *in the strong sense* over F_q , then $\text{Gen}(\text{Add}_n, \vec{\beta})$ is indeed secure over F_q as well where q is any prime power.

Theorem 3 (Diagonalization). *Let F_q be a finite field of order q . Let $\vec{\beta} = (\beta_1, \dots, \beta_n) \in (F_q^*)^n$ be multipliers and $\vec{\tau} = (\tau_1, \dots, \tau_n)$ be the leakage. Define*

$$\mu_i := \sum_{\vec{\ell} \in \Omega^n} \sum_{\zeta \in F_q^*} \left| \widehat{\mathbb{1}}_{\tau_i^{-1}(\ell_i)}(\zeta) \right|^n.$$

Then,

$$\varepsilon(\text{Gen}(\text{Add}_n, \vec{\beta})) \leq \prod_{i=1}^n \mu_i^{1/n} \leq \frac{1}{n} \sum_{i=1}^n \mu_i \leq \max(\mu_1, \dots, \mu_n)$$

Proof of Theorem 3. Let $D := \langle (\beta_1, \dots, \beta_n) \rangle$ be the vector space spanned by $(\beta_1, \dots, \beta_n)$. Then by Poisson summation formula,

$$\begin{aligned} 2 \cdot \text{SD}(\vec{\tau}(s), \vec{\tau}(U_F)) &\leq \sum_{\vec{\ell} \in \Omega^n} \sum_{\vec{\alpha} \in D^\perp} \prod_{i=1}^n \left| \widehat{\mathbb{1}}_{\tau_i^{-1}(\ell_i)}(\alpha_i) \right| && \text{(by triangle inequality)} \\ &= \sum_{\vec{\ell} \in \Omega^n} \sum_{\alpha \in F_q^*} \prod_{i=1}^n \left| \widehat{\mathbb{1}}_{\tau_i^{-1}(\ell_i)}(\alpha \beta_i) \right| \\ &\leq \prod_{i=1}^n \left(\sum_{\vec{\ell} \in \Omega^n} \sum_{\zeta \in F_q^*} \left| \widehat{\mathbb{1}}_{\tau_i^{-1}(\ell_i)}(\zeta) \right|^n \right)^{1/n} && \text{(by Hölder (Lemma 3))} \\ &\leq \frac{1}{n} \sum_{i=1}^n \sum_{\vec{\ell} \in \Omega^n} \sum_{\zeta \in F_q^*} \left| \widehat{\mathbb{1}}_{\tau_i^{-1}(\ell_i)}(\zeta) \right|^n && \text{(by AM-GM)} \end{aligned}$$

□

Note that when $\Omega = \{0, 1\}$, $\mu_i = \sum_{\ell \in \Omega^n} \sum_{\zeta \in \mathbb{F}_q^*} \prod_{j=1}^n \left| \widehat{\mathbb{1}}_{\tau_i^{-1}(\ell_j)}(\zeta) \right|$, which serves as an upper bound on the insecurity of additive secret sharing under the leakage $\vec{\tau}_i = (\tau_i, \dots, \tau_i)$, where $i \in [n]$.

If we have a concrete description of the Fourier coefficients of the leakage function, this can be used to obtain tight bounds on the ℓ^n norm, which in turn yields strong security guarantees. Even in settings where a full ℓ^n norm estimation is unavailable, a bound on the second-largest Fourier coefficient is often sufficient to establish security bounds for $\text{Gen}(\text{Add}_n, \vec{\beta})$. In particular, by leveraging bounds on the second-largest coefficient, we can derive several results in the following generalized setting.

Let $\text{Lin}_{n,k'}$ denote a threshold linear secret sharing scheme over F_q with n parties and reconstruction threshold k' , where any subset of k' or more parties can uniquely reconstruct the secret, and any subset with fewer than k' parties gains no information. We define $\text{Gen}(\text{Lin}_{n,k'}, \vec{\beta})$ for $\vec{\beta} \in (F_q^*)^n$ analogously to Definition 2: the scheme first generates shares (s'_1, \dots, s'_n) of a secret s , and then post-processes them into $(\beta_1^{-1} s'_1, \dots, \beta_n^{-1} s'_n)$. One construction of $\text{Lin}_{n,k'}$ is via Massey's secret sharing [Mas95] instantiated with an $[n+1, k'+1]$ -MDS linear code. In this case, the shares of secret 0 form a $(k'+1)$ -dimensional vector space $C_0 \subseteq F_q^{n+1}$, and the shares of any nonzero secret s form an affine shift of C_0 (the $+1$ in the dimensions accounts for the entry reserved for the secret). See [MPSW21] for a concise treatment of this concept.

Imported Theorem 1 (Benhamouda et al. [BDIR18, BDIR21] restated). Let $\mathcal{S}_{n,k'}$ be a linear n -party secret sharing scheme over \mathbb{F} whose shares of 0 form a vector space of dimension k' of \mathbb{F}^n . Consider $\tau_i: \mathbb{F} \rightarrow R_i$ for $i \in \{1, 2, \dots, n\}$. Partition the indices into $[n] = I_1 \cup I_2 \cup I_3$, where I_1, I_2 each of size $n - k'$. Let, $D = \{\{\zeta_i\}_{i \in I_3} : \vec{\zeta} \in \mathcal{S}_{n,k'}^\perp\}$. Then for any arbitrary $\vec{\beta} \in (\mathbb{F}^*)^n$, the following bound holds.

$$\begin{aligned} &\varepsilon(\text{Gen}(\mathcal{S}_{n,k'}, \vec{\beta}), \vec{\tau}) \\ &\leq \left(\prod_{j \in I_1 \cup I_2} \sum_{\ell_i \in R_i} \|\mathbb{1}_{\tau_i^{-1}(\ell_i)}\|_2 \right) \cdot \left(\sum_{\{\ell_j \in R_j\}_{j \in I_3}} \max_{\vec{\zeta} \in D \setminus \{\vec{0}\}} \prod_{j \in I_3} \left| \widehat{\mathbb{1}}_{\tau_j^{-1}(\ell_j)}(\beta_j \zeta_j) \right| \right) \end{aligned}$$

From [Imported Theorem 1](#) and using the result in [\[FMM⁺24\]](#), we could obtain the following corollary which is an extension of the result in [\[FMM⁺24\]](#) to generalized linear secret sharing scheme against hamming weight leakage attack.

Corollary 2. *Let $\text{Lin}_{n,k'}$ be a linear secret sharing and $\vec{\tau} = \vec{\text{wt}}$ where wt is the Hamming weight leakage function. Then for $\mathbb{F} = \mathbb{F}_p$, where p is Mersenne, the following bound holds.*

$$\varepsilon\left(\text{Gen}(\text{Lin}_{n,k'}; \vec{\beta}), \vec{\text{wt}}\right) \leq \mathcal{O}\left(\lambda^{\frac{5n-6k'-2}{4}}\right)$$

in particular, for $k' \geq \frac{5n}{6} + c$, $\text{Gen}(\text{Lin}_{n,k'}; \vec{\beta})$ is $\mathcal{O}(\lambda^{-c})$ -insecure against hamming weight leakage attack.

Moreover, using the results in [\[BS92\]](#) and [Imported Theorem 1](#), we can obtain the following corollary, which establishes the security of the generalized linear secret sharing scheme against any arbitrary low-range leakage function over a binary extension field, where the the leakage function's output is computed via a polynomial over the input bits.

Corollary 3. *Let $\text{Lin}_{n,k'}$ be a linear secret sharing and let $\vec{\tau} = (\tau_1, \tau_2, \dots, \tau_n)$ where each $\tau_i : \mathbb{F}_{2^\lambda} \rightarrow R_i$ is some leakage function defined as follows. For every $i \in [n]$, $\tau_i(x) = P_i(\vec{x}) = P_i(x_1, x_2, \dots, x_\lambda)$ for some polynomial $P_i : \mathbb{F}_{2^\lambda} \rightarrow R_i$, and $\vec{x} = (x_1, x_2, \dots, x_\lambda)$ denotes the binary vector representation of $x \in \mathbb{F}_{2^\lambda}$ over \mathbb{F}_2 . Let, $m = \max_i \log(\text{card}(R_i))$, then we have the following bound.*

$$\varepsilon\left(\text{Gen}(\mathcal{S}_{n,k'}; \vec{\beta}), \vec{\tau}\right) \leq \mathcal{O}\left(2^{\frac{mn}{2}} \cdot (\text{poly}(\lambda))^{-(3k'-2n+1)}\right)$$

in particular, for some constant $m \geq 1$ and $k' \geq \frac{2n}{3} + c$, $\text{Gen}(\mathcal{S}_{n,k'}; \vec{\beta})$ is $\mathcal{O}(\lambda^{-c})$ -insecure against the leakage function $\vec{\tau}$.

4 Open Problems

Following are the immediate open problem in light of our work.

1. *$n = 2$ parties.* Our recipe of using the Fourier proxy hits a natural bottleneck when $n = 2$. Even using the most optimistic estimates of Krawtchouk polynomial evaluations, [Appendix C.2](#) demonstrates that our approach cannot prove the security for $n = 2$ case. New technical machinery is required for this case.
2. *Attacks.* We proved that if our minimum score of $\vec{\beta}$ is large then it is sufficient to prove the security of the scheme. Is high minimum score also necessary? More concretely, given a vector of multipliers $\vec{\beta}$, does the insecurity of $\text{Gen}(\text{Add}_n, \vec{\beta})$ surpass a specific insecurity budget ε ?
3. *Prime modulus.* The case of Hamming weight leakage for Mersenne prime modulus was explored by Faust et al. [\[FMM⁺24\]](#) when $n \geq 5$. The cases of general primes and, even for Mersenne primes, $n \in \{2, 3, 4\}$ remains open.

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Supporting Materials.

A Lemmas for the Analysis

Lemma 1 (Chernoff Bound [McD98, Theorem 2.1]). *Let $X_1, X_2, \dots, X_\lambda$ be independent Bernoulli random variables with $\mathbf{E}(X_i) = p$ for each i . Then for any $t \geq 0$,*

$$\Pr\left(\left|\sum_{i=1}^{\lambda} X_i - \lambda p\right| \geq \lambda t\right) \leq 2 \exp(-2\lambda t^2).$$

In particular, for $t = \lambda^{-\frac{1}{2} + \tau}$, the upper bound is $2 \exp(-2\lambda^{2\tau})$.

Lemma 2 (Poisson Summation Formula [O'D21, Chapter 3.3]). *Let $C \subseteq F^n$ denote the set of all shares of the secret $0 \in F = F_{p^\lambda}$. Let $\vec{v} \in F^n$ denote an arbitrary secret share of the secret $1 \in F$. Consider an arbitrary local leakage $\vec{\tau}: F^n \rightarrow \Omega^n$. For any secret $s \in F$, let $\vec{\tau}(s)$ denote the distribution of the leakage $\vec{\tau}(\vec{x})$, where \vec{x} is sampled uniformly at random from the set $s \cdot \vec{v} + C$. The following identity holds for any leakage value $\vec{\ell} \in \Omega^n$.*

$$\Pr[\vec{\tau}(s) = \vec{\ell}] = \sum_{\vec{z} \in C^\perp} \left(\prod_{i=1}^n \widehat{\mathbb{1}}_{\tau_i^{-1}(\ell_i)}(z_i) \right) \cdot \chi_1(s \cdot \langle \vec{z}, \vec{v} \rangle). \quad (26)$$

where $\chi_1(x) = \omega_p^{\text{Tr}_{F/F_p}(x)}$, whose modulus is $|\chi_1(x)| = 1$ for any $x \in F$.

Lemma 3 (Generalized Hölder's Inequality [HLP52, Theorem 11]). *Let p_1, \dots, p_n be positive real numbers such that $p_1 + \dots + p_n = 1$, and $(x_i^{(1)})_{i=1}^k, (x_i^{(2)})_{i=1}^k, \dots, (x_i^{(n)})_{i=1}^k$ be sequences of real numbers. Then,*

$$\sum_{i=1}^k |x_i^{(1)}|^{p_1} \cdots |x_i^{(n)}|^{p_n} \leq \left(\sum_{i=1}^k |x_i^{(1)}| \right)^{p_1} \cdots \left(\sum_{i=1}^k |x_i^{(n)}| \right)^{p_n}$$

Lemma 4. *For $\lambda \in \{1, 2, \dots\}$, $w \in \{1, \dots, \lambda\}$, and $\zeta \in F$, we have $|\widehat{\mathbb{1}}_w(\zeta)| \leq B(\zeta)$ where*

$$B(\zeta) := \begin{cases} \lambda^{-1/2}, & \text{if } \text{wt}(\zeta) \in \{0, \lambda\} \\ \lambda^{-1}, & \text{if } \text{wt}(\zeta) \in \{1, \lambda - 1\} \\ 4 \cdot \lambda^{-3/2}, & \text{if } \text{wt}(\zeta) \in \{2, \lambda - 2\} \\ \lambda^{-\frac{1}{4}} \binom{\lambda}{\text{wt}(\zeta)}^{-\frac{1}{2}}, & \text{otherwise.} \end{cases}$$

And hence,

$$\sum_{\text{wt}(\zeta)=1}^{\lambda-1} (B(\zeta))^n \leq 3 \cdot 5^{\frac{5n}{2}-4} \cdot \lambda^{-n+1}$$

Lemma 5. *For any $\vec{\beta} = (\beta_1, \beta_2, \dots, \beta_n) \in (F^*)^n$ and $\vec{w} = (w_1, w_2, \dots, w_n) \in \{1, 2, \dots, \lambda\}^n$,*

$$\prod_{i=1}^n |\widehat{\mathbb{1}}_{w_i}(\beta_i \zeta)| \leq \lambda^{-\frac{n}{2}} \cdot \exp(-\text{Score}(\zeta; \vec{\beta}))$$

Proof of [Lemma 4](#) and [Lemma 5](#) can be found in [Appendix A.1](#).

A.1 Proof of Upper Bounds of Fourier Coefficients (Lemma 4 and Lemma 5)

Claim 1. For all $\lambda \in \{1, 2, \dots\}$, $w \in \{1, 2, \dots, \lambda\}$, and $\zeta \in F$,

$$\widehat{\mathbb{1}}_w(\zeta) = \frac{1}{2^\lambda} \sum_{k=0}^{\text{wt}(\zeta)} (-1)^k \binom{\text{wt}(\zeta)}{k} \binom{\lambda - \text{wt}(\zeta)}{w - k}$$

Proof of Claim 1. By definition of Fourier transformation,

$$\begin{aligned} \widehat{\mathbb{1}}_w(\zeta) &:= \frac{1}{2^\lambda} \sum_{x \in F} \mathbb{1}_w(x) \cdot (-1)^{\langle \zeta, x \rangle} = \frac{1}{2^\lambda} \sum_{\substack{x \in F \\ \text{wt}(x)=w}} (-1)^{\langle \zeta, x \rangle} \\ &= \frac{1}{2^\lambda} \sum_{k=0}^{\text{wt}(\zeta)} (-1)^k \cdot \#\{x: \langle \zeta, x \rangle = k\} \end{aligned}$$

Let us count the number of $x \in \text{wt}^{-1}(w) \subseteq F$ that satisfies $\langle \zeta, x \rangle = k$, given $k \in [0, \text{wt}(\zeta)]$. ζ has $\text{wt}(\zeta)$ -many 1's in its binary representation. In order for $x \in F$ to give $\langle \zeta, x \rangle = k$, it should have k -many overlapping 1's with ζ . Its remaining $(w - k)$ -many 1's can lie anywhere outside the digits where ζ has a 1 in it (because otherwise it will induce more overlapping 1's, making $\langle \zeta, x \rangle$ greater than k). Hence,

$$\#\{x: \langle \zeta, x \rangle = k\} = \binom{\text{wt}(\zeta)}{k} \binom{\lambda - \text{wt}(\zeta)}{w - k}$$

and therefore,

$$\widehat{\mathbb{1}}_w(\zeta) = \frac{1}{2^\lambda} \sum_{k=0}^{\text{wt}(\zeta)} (-1)^k \cdot \#\{x: \langle \zeta, x \rangle = k\} = \frac{1}{2^\lambda} \sum_{k=0}^{\text{wt}(\zeta)} (-1)^k \binom{\text{wt}(\zeta)}{k} \binom{\lambda - \text{wt}(\zeta)}{w - k}$$

as desired. \square

Remark 8. Observe from Claim 1 above that the value of $\widehat{\mathbb{1}}_w(\zeta)$ depends only on w and $\text{wt}(\zeta)$. In other words, $\widehat{\mathbb{1}}_w(\zeta)$ is a symmetric function in the sense that it is invariant to the permutation of digits of ζ . This is not an unexpected outcome because $\widehat{\mathbb{1}}_w(x)$ is a symmetric Boolean function itself, and Fourier transform of symmetric Boolean function over Boolean hypercube should also be symmetric (see [KLM⁺09, ST11, OWZ11], for more details). \triangle

Claim 2. For all $\lambda \in \{1, 2, \dots\}$, $w \in \{1, 2, \dots, \lambda\}$, and $\zeta \in F$,

$$\begin{aligned} \left| \widehat{\mathbb{1}}_w(\zeta) \right| &\leq \frac{1}{2^{\lambda/2}} \binom{\lambda}{w}^{1/2} \binom{\lambda}{\text{wt}(\zeta)}^{-1/2} \leq \frac{1}{\lambda^{1/4}} \binom{\lambda}{\text{wt}(\zeta)}^{-\frac{1}{2}} && \text{if } \zeta \in F \setminus \{0, \zeta^*\} \\ \left| \widehat{\mathbb{1}}_w(\zeta) \right| &= \frac{1}{2^\lambda} \binom{\lambda}{w} \leq \frac{1}{\sqrt{\lambda}} && \text{otherwise} \end{aligned}$$

Proof of Claim 2. Binomial coefficients are upper bounded by the central binomial coefficient, which can be then upper bounded as follows: for all $w \in \{1, 2, \dots, \lambda\}$,

$$\binom{\lambda}{w} \leq \binom{\lambda}{\lfloor \lambda/2 \rfloor} \leq \frac{2^\lambda}{\sqrt{\pi \lambda/2}} \leq \frac{2^\lambda}{\sqrt{\lambda}} \quad (27)$$

This immediately implies the required upper bound for $\zeta = 0$:

$$\widehat{\mathbb{1}}_w(\zeta) = \frac{1}{2^\lambda} \binom{\lambda}{w} \leq \frac{1}{\sqrt{\lambda}}$$

Similarly for $\zeta = \zeta^*$, we obtain the required upper bound using the following identity.

$$\left| \widehat{\mathbb{1}}_w(\zeta^*) \right| = \frac{1}{\text{card}(F)} \left| \sum_{x \in F} \mathbb{1}_w(x) \cdot (-1)^{\text{wt}(x)} \right| = \frac{1}{\text{card}(F)} \sum_{x \in F} \mathbb{1}_w(x) = \widehat{\mathbb{1}}_w(0)$$

Now consider an arbitrary $\zeta \in F^*$. Note that Parseval's identity ([Fact 1](#)) states

$$\sum_{\alpha \in F} \widehat{\mathbb{1}}_w(\alpha)^2 = \frac{1}{\text{card}(F)} \sum_{x \in F} \mathbb{1}_w(x)^2 \tag{28}$$

Simplifying the left-hand side of [Equation 28](#), we get

$$\frac{1}{\text{card}(F)} \sum_{x \in F} \mathbb{1}_w(x)^2 = \frac{1}{\text{card}(F)} \sum_{x \in F} \mathbb{1}_w(x) = \frac{1}{2^\lambda} \binom{\lambda}{w}$$

For the right-hand side of [Equation 28](#), note that, for any $\zeta \in F^*$,

$$\begin{aligned} \sum_{\alpha \in F} \widehat{\mathbb{1}}_w(\alpha)^2 &= \sum_{\alpha \in F} \left| \widehat{\mathbb{1}}_w(\alpha) \right|^2 \geq \sum_{\substack{\alpha \in F \text{ s.t.} \\ \text{wt}(\alpha) = \text{wt}(\zeta)}} \left| \widehat{\mathbb{1}}_w(\alpha) \right|^2 \\ &= \sum_{\substack{\alpha \in F \text{ s.t.} \\ \text{wt}(\alpha) = \text{wt}(\zeta)}} \left| \widehat{\mathbb{1}}_w(\zeta) \right|^2 = \binom{\lambda}{\text{wt}(\zeta)} \left| \widehat{\mathbb{1}}_w(\zeta) \right|^2 \end{aligned}$$

where the last equality comes from the observation made in [Remark 8](#). Putting these into [Equation 28](#) then gives us the following:

$$\begin{aligned} \sum_{\alpha \in F} \widehat{\mathbb{1}}_w(\alpha)^2 &= \frac{1}{\text{card}(F)} \sum_{x \in F} \mathbb{1}_w(x)^2 \\ \implies \frac{1}{2^\lambda} \binom{\lambda}{w} &\geq \binom{\lambda}{\text{wt}(\zeta)} \left| \widehat{\mathbb{1}}_w(\zeta) \right|^2 \\ \implies \left| \widehat{\mathbb{1}}_w(\zeta) \right| &\leq \frac{1}{2^{\lambda/2}} \binom{\lambda}{w}^{1/2} \binom{\lambda}{\text{wt}(\zeta)}^{-1/2} \\ \implies \left| \widehat{\mathbb{1}}_w(\zeta) \right| &\leq \frac{1}{\lambda^{1/4}} \binom{\lambda}{\text{wt}(\zeta)}^{-1/2} \tag{by [Equation 27](#)} \end{aligned}$$

as desired. □

Lemma 4. For $\lambda \in \{1, 2, \dots\}$, $w \in \{1, \dots, \lambda\}$, and $\zeta \in F$, we have $\left| \widehat{\mathbb{1}}_w(\zeta) \right| \leq B(\zeta)$ where

$$B(\zeta) := \begin{cases} \lambda^{-1/2}, & \text{if } \text{wt}(\zeta) \in \{0, \lambda\} \\ \lambda^{-1}, & \text{if } \text{wt}(\zeta) \in \{1, \lambda - 1\} \\ 4 \cdot \lambda^{-3/2}, & \text{if } \text{wt}(\zeta) \in \{2, \lambda - 2\} \\ \lambda^{-1/4} \binom{\lambda}{\text{wt}(\zeta)}^{-1/2}, & \text{otherwise.} \end{cases}$$

And hence,

$$\sum_{\text{wt}(\zeta)=1}^{\lambda-1} (B(\zeta))^n \leq 3 \cdot 5^{\frac{5n}{2}-4} \cdot \lambda^{-n+1}$$

Proof of Lemma 4. For brevity let us denote $\text{wt}^*(\zeta) := \min\{\text{wt}(\zeta), \lambda - \text{wt}(\zeta)\}$. The cases where $\text{wt}^*(\zeta) = 0$ and $\text{wt}^*(\zeta) \geq 2$ follow from Claim 2. Let us prove for the case $\text{wt}^*(\zeta) = 1$. Recall from Claim 1 that if $\text{wt}(\zeta) = 1$,

$$\begin{aligned} 2^\lambda \cdot \widehat{\mathbb{1}}_w(\zeta) &= \sum_{k=0}^1 (-1)^k \binom{1}{k} \binom{\lambda-1}{w-k} = \binom{\lambda-1}{w} - \binom{\lambda-1}{w-1} = \binom{\lambda}{w} \left(\frac{\lambda-2w}{\lambda} \right) \\ \implies \left| \widehat{\mathbb{1}}_w(\zeta) \right| &\leq \sqrt{\frac{2}{\pi\lambda}} \cdot \exp\left(-\frac{2 \cdot (\lambda/2 - w)^2}{\lambda+1}\right) \cdot \frac{|\lambda-2w|}{\lambda} && \text{(by Corollary 4)} \\ &\leq \sqrt{\frac{4}{\pi e}} \cdot \frac{1}{\lambda} \leq \frac{1}{\lambda} && \text{(by Corollary 5)} \end{aligned}$$

Then, we have

$$\sum_{\text{wt}^*(\zeta)=1} (B(\zeta))^n = 2 \cdot \binom{\lambda}{1} \cdot \left(\frac{1}{\lambda}\right)^n \leq 2 \cdot \lambda^{-n+1}$$

Similarly, for $\text{wt}^*(\zeta) = 2$ case,

$$\begin{aligned} 2^\lambda \cdot \widehat{\mathbb{1}}_w(\zeta) &= \binom{\lambda}{w} \frac{(\lambda-2w)^2 - \lambda}{\lambda(\lambda-1)} \\ \implies \left| \widehat{\mathbb{1}}_w(\zeta) \right| &\leq \sqrt{\frac{2}{\pi\lambda}} \cdot \exp\left(-\frac{2 \cdot (\lambda/2 - w)^2}{\lambda+1}\right) \cdot \frac{|(\lambda-2w)^2 - \lambda|}{\lambda(\lambda-1)} && \text{(by Corollary 4)} \\ &\leq \sqrt{\frac{2}{\pi}} \cdot \left(\frac{6}{e} + 2\right) \cdot \frac{1}{\lambda^{3/2}} \leq 4 \cdot \lambda^{-3/2} && \text{(by Corollary 5)} \end{aligned}$$

and we get

$$\sum_{\text{wt}^*(\zeta)=2} (B(\zeta))^n = 2 \cdot \binom{\lambda}{2} \cdot \left(4 \cdot \lambda^{-3/2}\right)^n \leq 4^n \cdot \lambda^{-\frac{3n}{2}+2}$$

For the remaining parts of the sum of $(B(\zeta))^n$, consider the following.

$$\begin{aligned}
& \sum_{\text{wt}(\zeta)=3}^{\lambda-2} (B(\zeta))^n \\
&= 2 \cdot \sum_{\text{wt}^*(\zeta) \geq 3} (B(\zeta))^n \\
&= 2 \cdot \lambda^{-\frac{n}{4}} \cdot \binom{\lambda}{3}^{-\frac{n}{2}+1} + 2 \cdot \lambda^{-\frac{n}{4}} \cdot \binom{\lambda}{4}^{-\frac{n}{2}+1} + 2 \cdot \sum_{k=5}^{\lambda/2} \lambda^{-\frac{n}{4}} \cdot \binom{\lambda}{k}^{-\frac{n}{2}+1} \\
&\leq 2 \cdot \lambda^{-\frac{n}{4}} \cdot \binom{\lambda}{3}^{-\frac{n}{2}+1} + 2 \cdot \lambda^{-\frac{n}{4}} \cdot \binom{\lambda}{4}^{-\frac{n}{2}+1} + 2 \cdot \frac{\lambda}{2} \cdot \lambda^{-\frac{n}{4}} \cdot \binom{\lambda}{5}^{-\frac{n}{2}+1} \\
&\leq 3^{\frac{3n}{2}-2} \cdot \lambda^{-n/4} \cdot \lambda^{-\frac{3n}{2}+3} + 4^{2n-3} \cdot \lambda^{-n/4} \cdot \lambda^{-2n+4} + 5^{\frac{5n}{2}-5} \cdot \lambda^{-n/4+1} \cdot \lambda^{-\frac{5n}{2}+5} \\
&= 3^{\frac{3n}{2}-2} \cdot \lambda^{-\frac{7n}{4}+3} + 4^{2n-2} \cdot \lambda^{-\frac{9n}{4}+4} + 5^{\frac{5n}{2}-5} \cdot \lambda^{-\frac{11n}{4}+6} \\
&\leq 3 \cdot 5^{\frac{5n}{2}-5} \cdot \lambda^{-\frac{7n}{4}+3} \quad \left(\text{as } \frac{7n}{4} - 3 < \frac{9n}{4} - 4 \text{ and } \frac{7n}{4} - 3 \leq \frac{11n}{4} - 6 \text{ for } n \geq 3 \right) \\
&\leq 5^{\frac{5n}{2}-4} \cdot \lambda^{-\frac{7n}{4}+3}
\end{aligned}$$

Therefore, comparing it with the previous expression, we obtain:

$$\begin{aligned}
\sum_{\text{wt}(\zeta)=1}^{\lambda-1} (B(\zeta))^n &\leq \underbrace{2 \cdot \lambda^{-n+1}}_{\text{Signal}} + \underbrace{4^n \cdot \lambda^{-\frac{3n}{2}+2}}_{\text{Median}} + \underbrace{5^{\frac{5n}{2}-4} \cdot \lambda^{-\frac{7n}{4}+3}}_{\text{Noise}} \\
&\leq 3 \cdot 5^{\frac{5n}{2}-4} \cdot \lambda^{-n+1} \quad \left(\text{because } n-1 < \frac{3n}{2} - 2 \text{ and } n-1 < \frac{7n}{4} - 3 \text{ for } n \geq 3 \right)
\end{aligned}$$

as desired. \square

From [Lemma 4](#), note that

$$\begin{aligned}
\left(2\lambda^{1/2+\tau} \right)^n \sum_{\zeta \in F \setminus \{0, \zeta^*\}} B(\zeta)^n &= 2^n \cdot \lambda^{n/2+n\tau} \cdot \sum_{\text{wt}(\zeta)=1}^{\lambda-1} (B(\zeta))^n \\
&\leq 3 \cdot 5^{\frac{5n}{2}-4} \cdot (2\lambda^\tau)^n \cdot \lambda^{-\frac{n}{2}+1} \\
&\leq 2^n \cdot 5^{\frac{5n}{2}-3} \cdot \lambda^{\tau n} \cdot \lambda^{-\frac{n}{2}+1}
\end{aligned}$$

which is small for sufficiently large λ when $n \geq 3$.

Lemma 5. For any $\vec{\beta} = (\beta_1, \beta_2, \dots, \beta_n) \in (F^*)^n$ and $\vec{w} = (w_1, w_2, \dots, w_n) \in \{1, 2, \dots, \lambda\}^n$,

$$\prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right| \leq \lambda^{-\frac{n}{2}} \cdot \exp\left(-\text{Score}(\zeta; \vec{\beta})\right)$$

Proof of Lemma 5. If $\text{wt}(\beta_i \zeta) \notin \{0, \lambda\}$, by [Lemma 4](#),

$$\begin{aligned}
\left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right| &\leq \frac{1}{\lambda^{1/4}} \binom{\lambda}{\text{wt}(\beta_i \zeta)}^{-1/2} \\
&= \frac{1}{\lambda^{1/2}} \cdot \frac{1}{\lambda^{-1/4}} \binom{\lambda}{\text{wt}(\beta_i \zeta)}^{-1/2} \\
&= \lambda^{-1/2} \cdot \exp\left(-\frac{1}{2} \log \binom{\lambda}{\text{wt}(\beta_i \zeta)} + \frac{\log \lambda}{4}\right)
\end{aligned}$$

Otherwise (if $\text{wt}(\beta_i\zeta) \in \{0, \lambda\}$), we have $|\widehat{\mathbb{1}}_{w_i}(\beta_i\zeta)| \leq \lambda^{-1/2}$, and therefore

$$\begin{aligned}
& \prod_{i=1}^n |\widehat{\mathbb{1}}_{w_i}(\beta_i\zeta)| \\
&= \left(\prod_{i: \text{wt}(\beta_i\zeta) \in \{0, \lambda\}} |\widehat{\mathbb{1}}_{w_i}(\beta_i\zeta)| \right) \cdot \left(\prod_{i: \text{wt}(\beta_i\zeta) \notin \{0, \lambda\}} |\widehat{\mathbb{1}}_{w_i}(\beta_i\zeta)| \right) \\
&\leq \lambda^{-\frac{h^*}{2}} \cdot \left(\prod_{i: \text{wt}(\beta_i\zeta) \notin \{0, \lambda\}} \lambda^{-\frac{1}{2}} \cdot \exp\left(-\frac{1}{2} \log\left(\frac{\lambda}{\text{wt}(\beta_i\zeta)}\right) + \frac{\log \lambda}{4}\right) \right) \\
&\hspace{20em} \text{(Set } h^* := \#\{i: \text{wt}(\beta_i\zeta) \in \{0, \lambda\}\}) \\
&= \lambda^{-\frac{h^*}{2}} \cdot \lambda^{-\frac{(n-h^*)}{2}} \cdot \exp\left(-\sum_{i: \text{wt}(\beta_i\zeta) \notin \{0, \lambda\}} \left(\frac{1}{2} \log\left(\frac{\lambda}{\text{wt}(\beta_i\zeta)}\right) - \frac{\log \lambda}{4}\right)\right) \\
&= \lambda^{-\frac{n}{2}} \cdot \exp\left(-\sum_{i=1}^n \sigma(\text{wt}(\beta_i\zeta))\right) \\
&= \lambda^{-\frac{n}{2}} \cdot \exp\left(-\text{Score}(\zeta; \vec{\beta})\right)
\end{aligned}$$

which proves the lemma. □

B Binomial Coefficients Estimations

We aim to prove a de Moivre-Laplace form (a.k.a., Gaussian-looking) upper bound on the binomial coefficients similar to Agievich [Agi22] and Pain [Pai24], which will simplify our analysis later.

Corollary 4 (Binomial Coefficient Estimation). *For any $a \in \{1, 2, \dots\}$ and $b \in \{0, 1, \dots, a\}$, the following bound holds.*

$$\binom{a}{b} \leq \frac{2^a}{\sqrt{\pi \cdot (a/2)}} \cdot \exp\left(-2 \cdot \frac{(b - a/2)^2}{a + 1}\right)$$

This result will follow straightforwardly from Lemma 6 and Lemma 7 that prove the result for even and odd a , respectively.^[3] Chernoff bound immediately yields the bound $\binom{a}{b} \leq 2^a \cdot \exp\left(-2 \cdot \frac{(b - a/2)^2}{a}\right)$; our upper bound is tighter by a multiplicative factor of $\mathcal{O}(a^{-1/2})$.

Lemma 6. *For $n \in \{1, 2, \dots\}$ and $x \in \{0, 1, \dots, n\}$ the following bounds hold*

$$\binom{2n}{n-x} \leq \binom{2n}{n} \cdot \exp\left(-\frac{x^2}{n+1/2}\right) \leq \frac{2^{2n}}{\sqrt{\pi n}} \cdot \exp\left(-\frac{x^2}{n+1/2}\right)$$

By the Central Limit Theorem, we expect $\binom{2n}{n-x} \rightarrow \binom{2n}{n} \cdot \exp(-x^2/n)$, for fixed x/n as $n \rightarrow \infty$. Berry-Esseen [Tao12, Chapter 2.2] and Camp-Paulson [JKK05, Chapter 3.6], for example, additively bound the gap between these two distributions. This lemma will prove a de Moivre-Laplace form upper bound instead.

^[3]This result also translates into a lower bound for Euler's Beta function, c.f. [GM15], improving the lower bound when the two input parameters to Euler's Beta function are close.

Proof of Lemma 6. We will use the following fact for the proof of our lemma:

Claim 3. For $x \in (0, 1]$, we have $x \leq \exp\left(-2 \cdot \frac{1-x}{1+x}\right)$.

Proof. Substituting $t = 1 - x$, the inequality is equivalent to

$$\ln(1 - t) \leq -\frac{t}{1 - t/2} = \sum_{i \geq 1} -\frac{t^i}{2^{i-1}},$$

which is true by inspection. □

[Top07] presents tighter bounds. Our upper bound is equivalent to lower bounding $\ln(1 + x)$, for $x \in [0, \infty)$. The bound above corresponds to the lower bound ϕ_1 in [Top07, Table 1]. In general, such bounds are a consequence of the identity [Top07, Equation 28].

Now, for the proof of our lemma, consider the following manipulation:

$$\begin{aligned} \binom{2n}{n-x} &= \binom{2n}{n} \cdot \frac{(n-x+1) \cdots n}{(n+1) \cdots (n+x)} \\ &= \binom{2n}{n} \cdot \prod_{i=1}^x \frac{n-x+i}{n+x-i+1} && \text{(rearranging)} \\ &\leq \binom{2n}{n} \cdot \prod_{i=1}^x \exp\left(-2 \cdot \frac{2x-2i+1}{2n+1}\right) && \text{(using Claim 3)} \\ &= \binom{2n}{n} \cdot \exp\left(-2 \cdot \frac{2x^2 - x(x+1) + x}{2n+1}\right) \\ &= \binom{2n}{n} \cdot \exp\left(-\frac{x^2}{n+1/2}\right). \end{aligned}$$

The final part of the result follows from Fact 3 below.

Fact 3. For $n \in \{1, 2, \dots\}$, $\binom{2n}{n} \leq \frac{2^{2n}}{\sqrt{\pi n}}$.

Tighter estimates are possible (for example, [Sas99, Corollary 1]); however, for our application, this elementary estimate suffices. □

Lemma 7. For $n \in \{1, 2, \dots\}$ and $x \in \{0, 1, \dots, n\}$ the following bounds hold

$$\binom{2n+1}{n-x} \leq \binom{2n+1}{n} \cdot \exp\left(-\frac{x(x+1)}{n+1}\right) \leq \frac{2^{2n+1}}{\sqrt{\pi(n+1/2)}} \cdot \exp\left(-\frac{(x+1/2)^2}{n+1}\right)$$

Proof of Lemma 7. By similar reasoning as in the proof of Lemma 6,

$$\begin{aligned} &\binom{2n+1}{n-x} \\ &= \binom{2n+1}{n} \frac{(n-x+1) \cdots n}{(n+2) \cdots (n+x)} = \binom{2n+1}{n} \prod_{i=1}^x \frac{n-x+i}{n+x+2-i} \\ &\leq \binom{2n+1}{n} \prod_{i=1}^x \exp\left(-2 \cdot \frac{2(x+1-i)}{2n+2}\right) && \text{(by Claim 3)} \end{aligned}$$

$$\begin{aligned}
&= \binom{2n+1}{n} \exp\left(-\sum_{i=1}^x \frac{2(x+1-i)}{n+1}\right) \\
&= \binom{2n+1}{n} \exp\left(-\frac{x(x+1)}{n+1}\right) && \text{(proves the first part of the lemma)} \\
&= \frac{1}{2} \cdot \binom{2n+2}{n+1} \cdot \exp\left(-\frac{x(x+1)}{n+1}\right) \\
&\leq \frac{2^{2n+1}}{\sqrt{\pi(n+1)}} \cdot \exp\left(-\frac{x(x+1)}{n+1}\right) && \text{(by Fact 3)} \\
&= \frac{2^{2n+1}}{\sqrt{\pi(n+1)}} \cdot \exp\left(-\frac{(x+1/2)^2}{n+1}\right) \cdot \exp\left(\frac{1}{4(n+1)}\right) && \text{(completing the square)} \\
&= \frac{2^{2n+1}}{\sqrt{\pi(n+1/2)}} \cdot \exp\left(-\frac{(x+1/2)^2}{n+1}\right) \cdot \sqrt{\frac{n+1/2}{n+1}} \cdot \exp\left(\frac{1}{2(n+1)}\right) \\
&= \frac{2^{2n+1}}{\sqrt{\pi(n+1/2)}} \cdot \exp\left(-\frac{(x+1/2)^2}{n+1}\right) \cdot \sqrt{\left(1 - \frac{1}{2(n+1)}\right)} \cdot \exp\left(\frac{1}{2(n+1)}\right) \\
&\leq \frac{2^{2n+1}}{\sqrt{\pi(n+1/2)}} \cdot \exp\left(-\frac{(x+1/2)^2}{n+1}\right). && \text{(because } 1 - \theta \leq \exp(-\theta)\text{)}
\end{aligned}$$

This completes the proof of the second inequality. \square

C Limitations

C.1 Limitation of Approximating the Spectral Norm of the Hamming Slice via Hamming Ball

Let $\mathbb{1}_{\leq w}(x)$ be the indicator function for the Hamming ball of radius w : $\mathbb{1}_{\leq w}(x) = 1$ if $\text{wt}(x) \leq w$ and 0 otherwise. Then, the indicator function for Hamming slice can be written as $\mathbb{1}_w = \mathbb{1}_{\leq w} - \mathbb{1}_{\leq (w-1)}$, and by linearity of the Fourier transform we have $\widehat{\mathbb{1}}_w = \widehat{\mathbb{1}}_{\leq w} - \widehat{\mathbb{1}}_{\leq (w-1)}$, and hence by the triangle inequality, $|\widehat{\mathbb{1}}_w| \leq |\widehat{\mathbb{1}}_{\leq w}| + |\widehat{\mathbb{1}}_{\leq (w-1)}|$. In addition, the complement of $\mathbb{1}_{\leq w}$ is $\mathbb{1}_{>w}$, which is a *linear threshold function* (LTF).

Threshold functions are widely studied objects and with many of their Fourier-analytic properties already well-established. It is therefore natural to approximate the spectrum of the Hamming slice through such approaches. While this representation is precise, the resulting spectral estimates are too coarse for tight security analysis. Existing bounds for LTFs are asymptotic and do not provide meaningful guarantees in the Hamming slice regime (i.e., $\log \lambda$ -bit leakage function).

To see this, for $n \geq 3$, we consider the effect of approximating the central slice $\mathbb{1}_{\lambda/2}$ by the majority function $\mathbb{1}_{\geq \lambda/2}$. Denote the levelwise (level- k) ℓ_n -spectral weight as $W_{k,n}(f) := \sum_{|S|=k} |\widehat{f}(S)|^n$. Using classical results on LTFs [Bel91, Bou02, O'D21], one obtains

$$W_{k,n}(\mathbb{1}_{\geq \lambda/2}) = \mathcal{O}\left(k^{-3n/4} \cdot \binom{\lambda}{k}^{-n/2+1}\right)$$

As k increases, these terms decay rapidly, and the first few levels dominate the spectral weight, as shown in Figure 1. For sufficiently large λ , the total spectral weight ($0 < |S| < \lambda$) or the (almost) ℓ_n -spectral norm becomes $\mathcal{O}(\lambda^{-n/2+1})$. Applying this result to approximate the second summand

in our security analysis for $\mathbb{1}_{\lambda/2}$ yields an insecurity bound of $\mathcal{O}(\lambda)$, which is meaningless in our context.

In contrast, for the central Hamming slice $\mathbb{1}_{\lambda/2}$, using direct Krawtchouk analysis for the first few levels and Parseval's theorem for the rest, we prove that levelwise ℓ_n -spectral weights satisfy

$$W_{k,n}(\mathbb{1}_{\lambda/2}) = \begin{cases} \mathcal{O}\left(\lambda^{-n/2} \cdot \binom{\lambda}{k}^{-n/2+1}\right) & \text{for } k = 1, 2, 3, 4 \\ \mathcal{O}\left(\lambda^{-n/4} \cdot \binom{\lambda}{k}^{-n/2+1}\right) & \text{for } k \geq 5 \end{cases}$$

Note again that lower levels are the dominating levels as shown in [Figure 1](#). One can see then that for sufficiently large λ , the (almost) ℓ_n -spectral norm of $\mathbb{1}_{\lambda/2}$ is $\mathcal{O}(\lambda^{-n+1})$. Substituting this, we obtain the bound $\mathcal{O}(\lambda^{-n/2+1})$ for the second summand in our security analysis, which allows us to obtain polynomial security for all $n \geq 3$.

Therefore, the key distinction is that the naïve threshold-based estimate overestimates the low-level spectral weight of the Hamming slice, losing a critical $\lambda^{n/2}$ factor in the exponent of the spectral norm. Recovering this factor requires a direct spectral analysis using Krawtchouk polynomials, which accurately captures the insecurity.

C.2 Optimistic Analysis

We argue that, while retaining the technical framework of our analysis, using the most optimistic estimates of Krawtchouk evaluations, one can optimistically only hope to prove security of schemes for $n \geq 3$ parties. Given this observation, our presentation introduces only the minimum technical machinery to prove our security result for $n \geq 3$ parties. For $n = 2$, new analysis techniques need to be developed.

We begin with the observation that the insecurity is upper-bounded by

$$\sum_{\vec{w} \in \{0,1,\dots,\lambda\}^n} \sum_{\zeta \in F^*} \prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right|$$

In preparation for using the rearrangement lemma, we separate the quantity as:

$$\underbrace{\sum_{\vec{w} \in \{0,1,\dots,\lambda\}^n} \sum_{\zeta \in \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right|}_{\text{first summand}} + \underbrace{\sum_{\vec{w} \in \{0,1,\dots,\lambda\}^n} \sum_{\zeta \in F^* \setminus \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n \left| \widehat{\mathbb{1}}_{w_i}(\beta_i \zeta) \right|}_{\text{second summand}}$$

Upper bounding just the second summand non-trivially will be a challenge. Let $B(z; w)$ denote an upper bound on $\left| \widehat{\mathbb{1}}_w(\zeta) \right|$, where $z = \text{wt}(\zeta)$. Using this upper bound, we get the following upper bound on the second summand above.

$$\sum_{\vec{w} \in \{0,1,\dots,\lambda\}^n} \sum_{\zeta \in F^* \setminus \mathcal{S}_{\vec{\beta}}} \prod_{i=1}^n B(\text{wt}(\beta_i \zeta); w_i)$$

$B(z; w)$ will have the property that it will decrease as z gets closer to $\lambda/2$. When satisfying this property, we can use the rearrangement lemma to upper-bound the expression above and we

rearrange this expression.

$$\begin{aligned}
& \sum_{\vec{w} \in \{0,1,\dots,\lambda\}^n} \sum_{\zeta \in F \setminus \{0,\zeta^*\}} \prod_{i=1}^n B(\text{wt}(\zeta); w_i) \\
&= \sum_{\vec{w} \in \{0,1,\dots,\lambda\}^n} \sum_{1 \leq z \leq \lambda-1} \binom{\lambda}{z} \prod_{i=1}^n B(z; w_i) \\
&= \sum_{1 \leq z \leq \lambda-1} \binom{\lambda}{z} \cdot \left(\sum_{w \in \{0,1,\dots,\lambda\}} B(z; w) \right)^n \tag{29}
\end{aligned}$$

So, we need an estimate for $\sum_w B(z; w)$. $B(n; w)$ is an upper bound on the evaluation of the Krawtchouk polynomial. The most optimistic estimates from [Kra01, Theorem 10] put it as (roughly)^[4]

$$B(z; w)^2 = \lambda^{-1} \cdot \binom{\lambda}{w} 2^{-\lambda} \cdot \binom{\lambda}{z}^{-1}$$

We highlight that the multiplicative λ^{-1} factor is the non-trivial part; without that factor, the upper bound is straightforward. In light of this optimistic estimate, we have

$$\begin{aligned}
\sum_w B(z; w) &= \lambda^{-1/2} 2^{-\lambda/2} \binom{\lambda}{z}^{-1/2} \sum_w \binom{\lambda}{w}^{1/2} \\
&= \lambda^{-1/2} 2^{-\lambda/2} \binom{\lambda}{z}^{-1/2} \cdot 2^{\lambda/2} \lambda^{1/4}. \\
&= \lambda^{-1/4} \binom{\lambda}{z}^{-1/2}.
\end{aligned}$$

The last equality uses the asymptotic estimate from [GKP94, Answer to Problem 9.18 on page 593].^[5] We substitute this estimate back in Equation 29 to get our overall upper bound on the second summand.

$$\sum_{1 \leq z \leq \lambda-1} \binom{\lambda}{z}^{1-n/2} \cdot \lambda^{-n/4}.$$

For $n = 2$, the upper bound on the second summand is $\sqrt{\lambda}$, which is meaningless. Only for $n \geq 3$, the upper bound on the second summand can be meaningful. In fact, it suffices to (1) use an accurate estimate for $B(1; w)$ and (2) elsewhere use the trivial Parseval-based estimate

$$B(z; w)^2 = \binom{\lambda}{w} 2^{-\lambda} \binom{\lambda}{z}^{-1}.$$

^[4]The result of [Kra01, Theorem 10] is more nuanced. For back-of-the-envelope calculations, ignoring a small correction term, it implies $\binom{\lambda}{z} B(z; w)^2$ has order $(z(\lambda-z))^{-1/2} \cdot \binom{\lambda}{w} 2^{-\lambda}$. When $z = \Theta(\lambda)$, which is a constant fraction of all possible $z \in \{0, 1, \dots, \lambda\}$, this quantity is $\lambda^{-1} \cdot \binom{\lambda}{w} 2^{-\lambda}$.

^[5]An upper bound of $2^{\lambda/2} (\lambda+1)^{1/2}$ is straightforward using Cauchy-Schwartz. The tighter $\mathcal{O}(\lambda^{1/4})$ asymptotic term requires additional effort.

D Some Estimates: Sum of Powers of Binomial Coefficients

Claim 4. For arbitrary $\theta \in \mathbb{R}$ and $m \in \mathbb{R}_{>0}$, the following bound holds.

$$\sum_{\Delta \in \theta + \mathbb{Z}} \exp(-\Delta^2/m) \leq \sqrt{\pi m} + 1.$$

This upper bound admits an elementary proof for all m , which we present below. For large m , significantly tighter bounds could be derived by connecting this sum to the third Jacobi theta function. For example, [Zhu18, Theorem 2.2] proved the following estimate.

$$\left| \frac{1}{\sqrt{\pi m}} \sum_{\Delta \in \theta + \mathbb{Z}} \exp(-\Delta^2/m) - 1 \right| \leq \operatorname{csch}(\pi^2 m).$$

Proof of Claim 4. We will use the property that $\exp(-x^2/m)$ is decreasing for $x \geq 0$ and the fact that $\int_{-\infty}^{\infty} \exp(-t^2/m) dt = \sqrt{\pi m}$.

Because $\exp(-x^2/m)$ is even and the LHS is periodic in θ (with period 1), it suffices to consider $\theta \in [0, 1/2]$. For brevity, denote $f(\Delta) = \exp(-\Delta^2/m)$ and $\bar{\theta} = (1 - \theta)$. Then,

$$\begin{aligned} & \sum_{\Delta \in \theta + \mathbb{Z}} \exp(-\Delta^2/m) \\ &= \left(\theta f(\theta) + \sum_{\Delta \in \{1+\theta, 2+\theta, \dots\}} f(\Delta) \right) + \left(\bar{\theta} f(-\bar{\theta}) + \sum_{\Delta \in \{-\bar{\theta}-1, -\bar{\theta}-2, \dots\}} f(\Delta) \right) \\ & \quad + \bar{\theta} f(\theta) + \theta f(-\bar{\theta}) \\ &\leq \int_0^{\infty} f(t) dt + \int_{-\infty}^0 f(t) dt + \bar{\theta} \cdot f(0) + \theta \cdot f(0) \\ &\leq \sqrt{\pi m} + 1. \end{aligned}$$

This completes the proof of the claim. □

Remark 9. We can prove the tighter bound

$$\sum_{\Delta \in \theta + \mathbb{Z}} \exp(-\Delta^2/m) \leq \sqrt{\pi m} + \exp(-\hat{\theta}^2/m),$$

where $\hat{\theta} \in [0, 1/2]$ is the distance of θ from \mathbb{Z} , i.e., $\min_{i \in \mathbb{Z}} |\theta - i|$. △

Claim 5. For $m \in \{1, 2, \dots\}$, the following bound holds.

$$\sum_{i=0}^m \binom{m}{i}^{1/2} < \pi \cdot m^{1/4} \cdot 2^{m/2}.$$

Proof of Claim 5. Consider the following manipulation.

$$\begin{aligned} \sum_{i=0}^m \binom{m}{i}^{1/2} &\leq \frac{2^{m/2}}{(\pi m/2)^{1/4}} \sum_{i=0}^m \exp\left(-\frac{(i - m/2)^2}{m+1}\right) && \text{(using Corollary 4)} \\ &\leq \frac{2^{m/2}}{(\pi m/2)^{1/4}} \cdot \left(1 + \sqrt{\pi(m+1)}\right) && \text{(using Claim 4)} \\ &\leq 2^{m/2} \cdot \left(\max_{t \geq 1} \frac{1 + \sqrt{\pi(t+1)}}{(\pi t/2)^{1/4}} \cdot \frac{1}{t^{1/4}}\right) \cdot m^{1/4} \end{aligned}$$

The maximum is achieved at $t = 1$ and the maximum value is $(2/\pi)^{1/4} + 2 \cdot (\pi/2)^{1/4} < 3.14 < \pi$. \square

Remark 10. In general, for $p \in (0, 1]$, we can prove that

$$\begin{aligned} \sum_{i=0}^m \binom{m}{i}^p &\leq \frac{2^{pm}}{(\pi m/2)^{p/2}} \cdot \left(1 + \sqrt{\pi(m+1)/2p}\right) \\ &\leq \sqrt{2} \cdot (1 + \pi^{-1/2}) \cdot \frac{1}{\sqrt{p}} \cdot 2^{pm} \cdot m^{(1-p)/2} \end{aligned}$$

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E Concrete Upper Bound on Krawtchouk Polynomials

Claim 6. For $a \in \mathbb{R}_{>0}$ and $x, k \in \mathbb{R}_{\geq 0}$, the following bound holds

$$\exp(-x^2/a) \cdot x^k \leq \begin{cases} 1, & \text{if } k = 0. \\ (ka/2e)^{k/2}, & \text{otherwise.} \end{cases} \quad (30)$$

Proof. We first maximize $-\frac{x^2}{a} + k \cdot \ln x$. It is maximized at $x^2 = ka/2$. Therefore, $\exp(-x^2/a) \cdot x^k \leq \exp(-k/2) \cdot (ka/2)^{k/2}$. \square

The following estimates follow immediately from [Claim 6](#).

Corollary 5. For $x \in \mathbb{R}_{\geq 0}$ and $m \in \mathbb{Z}$, the following bounds hold.

1. For $m \geq 1$:

$$\exp\left(-\frac{2x^2}{m+1}\right) \cdot \frac{2x}{m} \leq \frac{2^{1/2}}{e^{1/2}} \cdot \frac{1}{m^{1/2}}.$$

2. For $m \geq 2$:

$$\exp\left(-\frac{2x^2}{m+1}\right) \cdot \frac{|(2x)^2 - m|}{m(m-1)} \leq \left(\frac{6}{e} + 2\right) \cdot \frac{1}{m}.$$

Proof. In this proof we will repeatedly use the bound

$$\exp\left(-\frac{2x^2}{m+1}\right) \cdot (2x)^k \leq \left(k \cdot \frac{m+1}{e}\right)^{k/2}, \quad (31)$$

which follows from [Claim 6](#) by setting $a = (m+1)/2$.

1. For $m \geq 1$,

$$\begin{aligned} \exp\left(-\frac{2x^2}{m+1}\right) \cdot \frac{2x}{m} &\leq \left(\frac{m+1}{e}\right)^{1/2} \cdot \frac{1}{m} && \text{(using Equation 31 with } k = 1) \\ &= \left(\frac{m+1}{em}\right)^{1/2} \cdot \frac{1}{m^{1/2}} \\ &\leq (2/e)^{1/2} \cdot \frac{1}{m^{1/2}}. && \text{(bound holds for } m \geq 1) \end{aligned}$$

2. For $m \geq 2$,

$$\begin{aligned}
& \exp\left(-\frac{2x^2}{m+1}\right) \cdot \frac{|(2x)^2 - m|}{m(m-1)} \\
& \leq \exp\left(-\frac{2x^2}{m+1}\right) \cdot \frac{(2x)^2 + m}{m(m-1)} \\
& = \exp\left(-\frac{2x^2}{m+1}\right) \cdot \frac{(2x)^2}{m(m-1)} + \exp\left(-\frac{2x^2}{m+1}\right) \cdot \frac{1}{(m-1)} \\
& \leq \left(2 \cdot \frac{m+1}{e}\right) \cdot \frac{1}{m(m-1)} + \frac{1}{m-1} \quad (\text{using Equation 31 with } k \in \{2, 0\}) \\
& = \left[\frac{2(m+1)}{e(m-1)} + \frac{m}{m-1}\right] \cdot \frac{1}{m} \\
& \leq \left(\frac{6}{e} + 2\right) \cdot \frac{1}{m} \quad (\text{bound holds for } m \geq 2.)
\end{aligned}$$

□

F Choice of Reconstruction Multipliers

Claim 7. Let $\Pi(Z) = Z^\lambda + Z + 1 \in F_2[Z]$ be irreducible, and put $F = F_{2^\lambda} = F_2[Z]/(\Pi(Z))$. Let $Z \in F$ be the primitive element with $Z^\lambda = Z + 1$. For

$$\zeta^* = 1 + Z + \dots + Z^{\lambda-1} = \frac{Z}{Z+1},$$

the Hamming weight of $Z^{\pm i}\zeta^*$ is

$$\text{wt}(Z^{\pm i}\zeta^*) = \begin{cases} \lambda - i, & 1 \leq i \leq \lambda - 2 \text{ for } Z^i\zeta^*, \\ \lambda - \frac{i}{2}, & 1 \leq i \leq \lambda - 1, i \text{ even, for } Z^{-i}\zeta^*, \\ \lambda - \frac{i+1}{2}, & 1 \leq i \leq \lambda - 1, i \text{ odd, for } Z^{-i}\zeta^*. \end{cases}$$

Proof. From $Z^\lambda = Z + 1$ we obtain

$$Z^i\zeta^* = 1 + Z^{i+1} + \dots + Z^{\lambda-1}, \quad 1 \leq i \leq \lambda - 2,$$

giving $\text{wt}(Z^i\zeta^*) = \lambda - i$. Also since $\Pi(Z) = 0$, we have (one can verify inductively)

$$Z^{-i} = Z^{\lambda-i} + Z^{\lambda-(i-1)} + \dots + Z^{\lambda-1} + 1.$$

If i is even this yields

$$Z^{-i}\zeta^* = (Z^{\lambda-(i-1)} + Z^{\lambda-(i-3)} + \dots + Z^{\lambda-1}) + \zeta^*,$$

so $\text{wt}(Z^{-i}\zeta^*) = \lambda - \frac{i}{2}$. If i is odd then

$$Z^{-i}\zeta^* = (Z^{\lambda-(i-1)} + Z^{\lambda-(i-3)} + \dots + Z^{\lambda-1}) + (Z^{\lambda-2} + Z^{\lambda-2} + \dots + Z),$$

giving $\text{wt}(Z^{-i}\zeta^*) = \lambda - \frac{i+1}{2}$. □

Claim 8. Let $\Pi(Z) = Z^\lambda + Z^j + 1 \in F_2[Z]$ be irreducible, and put $F = F_{2^\lambda} = F_2[Z]/(\Pi(Z))$. Let $Z \in F$ be the primitive element with $Z^\lambda = Z^j + 1$. Consider j to be sufficiently small compared to λ . For

$$\zeta^* = 1 + Z + \cdots + Z^{\lambda-1} = \frac{Z^j}{Z+1},$$

the Hamming weight of $Z^{\pm i}\zeta^*$ for $i \gg j$ is

$$\text{wt}(Z^{\pm i}\zeta^*) = \begin{cases} \lambda - i, & i \leq \lambda - j - 1 \text{ for } Z^i\zeta^*, \\ \lambda - \frac{kj}{2}, & i = kj \leq \lambda - 1, \text{ } k \text{ even, for } Z^{-i}\zeta^*, \\ \lambda - j - \frac{(k-1)j}{2} & i = kj \leq \lambda - j, \text{ } k \text{ odd, for } Z^{-i}\zeta^*. \end{cases}$$

Proof. From $Z^\lambda = Z^j + 1$ we obtain

$$Z^i\zeta^* = 1 + Z + Z^2 + \cdots + Z^{j-1} + Z^{j+i} + \cdots + Z^{\lambda-1}, \quad 1 \leq i \leq \lambda - j - 1,$$

giving $\text{wt}(Z^i\zeta^*) = \lambda - i$.

Also since $\Pi(Z) = 0$, for some $i = kj$ we have (one can verify inductively)

$$Z^{-i} = \sum_{l=0}^{k-1} Z^{\lambda-(k-l)j} + 1.$$

Now consider the following cases :

- k is even: Then $(\sum_{l=0}^{k-1} Z^{\lambda-(k-l)j}) \cdot Z^j = p(Z) \cdot (Z+1)$ for some $p(Z)$ with $\frac{kj}{2}$ many terms. This yields $Z^{-i}\zeta^* = Z^j \cdot p(Z) + \zeta^*$, and so $\text{wt}(Z^{-i}\zeta^*) = \lambda - \frac{kj}{2}$.
- If k is odd: $(\sum_{l=0}^{k-2} Z^{\lambda-(k-l)j}) \cdot Z^j = p(Z) \cdot (Z+1)$ for some polynomial $p(Z)$ starts with $Z^{\lambda-j-1}$ and ends with $Z^{\lambda-kj}$ with $\frac{(k-1)j}{2}$ many terms and $(Z^{\lambda-j} + 1) \cdot Z^j = (Z^{\lambda-1} + \cdots + Z^j) \cdot (Z+1)$. Since $\lambda \geq i + j$, $p(Z)$ lies between $Z^{\lambda-1}$ and Z^j . This yields $\text{wt}(Z^{-i}\zeta^*) = \lambda - j - \frac{(k-1)j}{2}$.

□

Claim 9. Let $\Pi(Z) = Z^\lambda + Z^j + 1 \in F_2[Z]$ be irreducible, and put $F = F_{2^\lambda} = F_2[Z]/(\Pi(Z))$. Let $Z \in F$ be the primitive element with $Z^\lambda = Z^j + 1$. For

$$\zeta^* = 1 + Z + \cdots + Z^{\lambda-1} = \frac{Z^j}{Z+1},$$

then $\text{wt}(Z^j\zeta^*) = \text{wt}(Z^{-j}\zeta^*) = \lambda - j$.

Proof. From $Z^\lambda = Z^j + 1$ we obtain

$$Z^j\zeta^* = 1 + Z + Z^2 + \cdots + Z^{j-1} + Z^{2j} + \cdots + Z^{\lambda-1},$$

giving $\text{wt}(Z^j\zeta^*) = \lambda - j$. On the other hand, we have

$$Z^{-j}\zeta^* = \frac{Z^\lambda + Z^j}{Z+1} = Z^{\lambda-1} + Z^{\lambda-2} \cdots + Z^j,$$

and so $\text{wt}(Z^{-j}\zeta^*) = \lambda - j$.

□

Claim 10. Let $\Pi(Z) = Z^\lambda + c_{\lambda-1}Z^{\lambda-1} + \cdots + c_1Z + 1 \in F[Z]$ be a monic irreducible polynomial, and let Z be a root of $p(Z)$. Consider the polynomial basis

$$\{1, Z, Z^2, \dots, Z^{\lambda-1}\}$$

of F_{2^λ} over F_2 , and define

$$S = 1 + Z + Z^2 + \cdots + Z^{\lambda-1}.$$

Then the Hamming weights of $Z \cdot S$ and $Z^{-1} \cdot S$ (in the polynomial basis) are equal:

$$\text{wt}(Z \cdot S) = \text{wt}(Z^{-1} \cdot S) = \lambda - \#\{i : c_i = 1\}$$

Proof. Since $\Pi(Z) = 0$, we have

$$Z^\lambda + c_{\lambda-1}Z^{\lambda-1} + \cdots + c_1Z + 1 = 0,$$

so that

$$Z^{-1} = Z^{\lambda-1} + c_{\lambda-1}Z^{\lambda-2} + \cdots + c_1.$$

Multiplying S by Z gives

$$Z \cdot S = 1 + (1 + c_1)Z + (1 + c_2)Z^2 + \cdots + (1 + c_{\lambda-1})Z^{\lambda-1}.$$

Similarly, multiplying S by Z^{-1} gives

$$Z^{-1} \cdot S = (1 + c_1) \cdot 1 + (1 + c_2)Z + \cdots + (1 + c_{\lambda-1})Z^{\lambda-2} + 1 \cdot Z^{\lambda-1}.$$

Thus, the coefficients of $Z \cdot S$ and $Z^{-1} \cdot S$ are identical up to a cyclic shift, which preserves the Hamming weight. Hence

$$\text{wt}(Z \cdot S) = \text{wt}(Z^{-1} \cdot S) = \lambda - \#\{i : c_i = 1\}.$$

□