CS314 Summer 2013

Midterm Exam Part 1

Name:____

Problem	Max.	Grade
1.	10	
2.	10	
3.	10	
4.	15	
5.	15	
Total:	60	

1. (10 pts) Write the decimal representation for the following floating point number in binary

10101111 10010100 00000000 00000000 00000000 00000000 00000000 00000000

Solution:

MAC-32 double number $\Rightarrow S=1,\ n=01011111001_2=761$

$q=01$

$$\Rightarrow \quad -1.01_2\times 2^{761-1023} = -1.01_2\times 2^{-262}$$

$$= -(1+2^{-2})_{10}\times 2^{-262}$$

$$= -1.25\times 2^{-262}$$

$$\approx -1.6867517\times 10^{-79}$$

2. (10 pts)Use the bisection method to solve e^x-4-2x=0, starting with the interval [-3,-1] Do 4 iterations

$k=1$ $\quad a=-3 \quad f_a=2.049781$

$b=-1 \quad f_b=-1.632121$

$c=-2 \quad f_c=0.135335$

$k=2$ $\quad a=-2 \quad f_a=0.135335$

$b=-1 \quad f_b=-1.632121$

$c=-1.5 \quad f_c=-0.776870$

$k=3$ $\quad a=-2 \quad f_a=0.135335$

$b=-1.5 \quad f_b=-0.776870$

$c=-1.75 \quad f_c=-0.326226$

$k=4$ $\quad a=-2 \quad f_a=0.135335$

$b=-1.75 \quad f_b=-0.326226$

$c=-1.875 \quad f_c=-0.096645$

$X_1=-1$

3. (10 pts.) Use the secant method to solve e^x-4-2x=0 starting at x0=-2. Also describe the type of convergence/divergence.

Secant method:

$$P_{k+1}=g(P_k,P_{k-1})=P_k-\frac{f(P_k)(P_k-P_{k-1})}{f(P_k)-f(P_{k-1})}$$

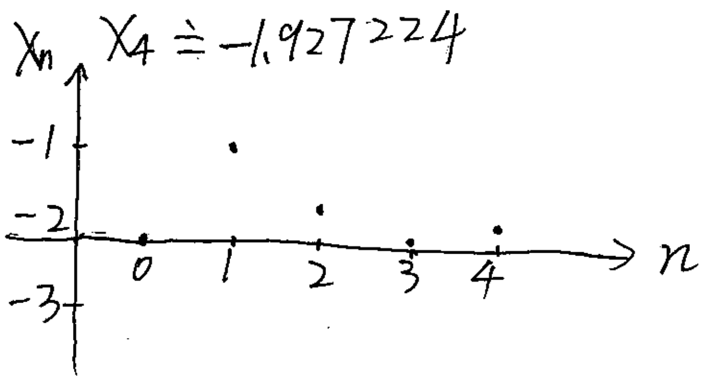
$X_0=-2$

$X_1=-1$

$$X_2=X_1-\frac{f(X_1)(X_1-X_0)}{f(X_1)-f(X_0)}\doteq-1.923429$$

$X_3\doteq-1.927428$

$X_4\doteq-1.927224$



oscillating convergence

4. (10 pts) Solve the following system of linear equations using Gaussian elimination.

$$
\begin{aligned}
2X_1 + 3X_2 + 1X_3 - 2X_4 &= 4 \\
5X_1 + 4X_2 - 1X_3 + 0X_4 &= 8 \\
0X_1 + 4X_2 + 3X_3 - 2X_4 &= 5 \\
5X_1 - 2X_2 + 2X_3 + 7X_4 &= 12
\end{aligned}
$$

$$
\begin{bmatrix} 2 & 3 & 1 & -2 \\ 5 & 4 & -1 & 0 \\ 0 & 4 & 3 & -2 \\ 5 & -2 & 2 & 7 \end{bmatrix}
\begin{bmatrix} 4 \\ 8 \\ 5 \\ 12 \end{bmatrix}
\Rightarrow
\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & -1 \\ 0 & 1 & \frac{3}{4} & -\frac{1}{2} \\ 0 & -\frac{7}{2} & -\frac{7}{2} & 5 \\ 0 & -\frac{19}{2} & -\frac{1}{2} & 12 \end{bmatrix}
\begin{bmatrix} 2 \\ \frac{5}{4} \\ -2 \\ 2 \end{bmatrix}
\Rightarrow
\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & -1 \\ 0 & 1 & \frac{3}{4} & -\frac{1}{2} \\ 0 & 0 & -\frac{7}{8} & \frac{13}{4} \\ 0 & 0 & \frac{53}{8} & \frac{29}{4} \end{bmatrix}
\begin{bmatrix} 2 \\ \frac{5}{4} \\ \frac{19}{8} \\ \frac{111}{8} \end{bmatrix}
$$

$$
\Rightarrow
\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & -1 \\ 0 & 1 & \frac{3}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{26}{7} \\ 0 & 0 & \frac{53}{8} & \frac{29}{4} \end{bmatrix}
\begin{bmatrix} 2 \\ \frac{5}{4} \\ -\frac{19}{7} \\ \frac{111}{8} \end{bmatrix}
\Rightarrow
\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & -1 \\ 0 & 1 & \frac{3}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{26}{7} \\ 0 & 0 & 0 & \frac{223}{7} \end{bmatrix}
\begin{bmatrix} 2 \\ \frac{5}{4} \\ -\frac{19}{7} \\ \frac{233}{7} \end{bmatrix}
$$

$$
\Rightarrow
\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & -1 \\ 0 & 1 & \frac{3}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{26}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 2 \\ \frac{5}{4} \\ -\frac{19}{7} \\ 1 \end{bmatrix}
\Rightarrow
\begin{bmatrix} X_1 = 1 \\ X_2 = 1 \\ X_3 = 1 \\ X_4 = 1 \end{bmatrix}
$$

5. (10 pts) Use the Gauss-Seidel iteration methods to find (x_k, y_k, z_k) for k=1,2,3. Start at the point (x_0, y_0, z_0)=0,0,0. You can change the order of the equations so the method can converge.

$$2x + 9y - 2z = 14$$

$$7x - y + 3z = 14$$

$$-x + 3y + 8z = 29$$

Rearrange the equations to make the coefficient matrix strictly diagonal dominant

$$\begin{cases} 7x - y + 3z = 14 \\ 2x + 9y - 2z = 14 \\ -x + 3y + 8z = 29 \end{cases} \Rightarrow \begin{cases} x = \dfrac{14 + y - 3z}{7} \\ y = \dfrac{14 - 2x + 2z}{9} \\ z = \dfrac{29 + x - 3y}{8} \end{cases} \Rightarrow \begin{cases} x_1 = \dfrac{14+0+0}{7} = 2 \\ y_1 = \dfrac{14-4+0}{9} \approx 1.1111 \\ z_1 = \dfrac{29+2-3.333}{8} \doteq 3.4583 \end{cases}$$

$$\Rightarrow \begin{cases} x_2 = \dfrac{14+1.11111-3\times 3.4583}{7} \doteq 0.67659 \\ y_2 = \dfrac{14-2\cdot 0.67659+2\cdot 3.4583}{9} \doteq 2.173714 \\ z_3 = \dfrac{29+0.67659-3\times 2.173714}{8} \doteq 2.894431 \end{cases} \Rightarrow \begin{cases} x_3 = \dfrac{14+2.173714-3\cdot 2.894431}{7} \doteq 1.07006 \\ y_3 = \dfrac{14-2\cdot 1.07006+2\cdot 2.894431}{9} \doteq 1.960971 \\ z_3 = \dfrac{29+1.07006-3\cdot 1.960971}{8} \doteq 2.898393 \end{cases}$$

$$\begin{cases} x_4 = \dfrac{14+1.960971-3\cdot 2.898393}{7} \doteq 1.037970 \\ y_4 = \dfrac{14-2\cdot 1.037970+2\cdot 2.898393}{9} \doteq 1.968983 \\ z_4 = \dfrac{29+1.037970-3\cdot 1.968983}{8} \doteq 3.016378 \end{cases}$$

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Midterm Exam Part 2

Name:____

Problem	Max.	Grade
6.	15	
7.	10	
8.	10	
9.	15	
Total:	50	

6. (15 pts.) Consider the nonlinear system

$$0 = 5x^2 - 2y - 1$$
$$0 = 3y^2 - x - 11$$

Start with (x0,y0)=(1.2,1.2) and apply Newton's method for systems of non-linear equations to compute (x1,y1) and (x2,y2).

Newton: $X_{k+1} = X_k - J(X_k)^{-1} F(X_k)$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x}, & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x}, & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 10x & -2 \\ -1 & 6y \end{bmatrix} \Rightarrow J^{-1} = \frac{1}{60xy-2} \begin{bmatrix} 6y & 2 \\ 1 & 10x \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1.2 \\ 1.2 \end{bmatrix} - \frac{1}{60 \cdot 1.2 \cdot 1.2 - 2} \begin{bmatrix} 6 \cdot 1.2 & 2 \\ 1 & 10 \cdot 1.2 \end{bmatrix} \begin{bmatrix} 5 \cdot 1.2^2 - 2 \cdot 1.2 - 1 \\ 3 \cdot 1.2^2 - 1.2 - 11 \end{bmatrix} \doteq \begin{bmatrix} 1.06256 \\ 2.27536 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1.06256 \\ 2.27536 \end{bmatrix} - \frac{1}{60 \cdot 1.06256 \cdot 2.27536} \begin{bmatrix} 6 \cdot 2.27536 & 2 \\ 1 & 10 \cdot 1.06256 \end{bmatrix} \begin{bmatrix} 5 \cdot 1.06256^2 - 2 \cdot 2.27536 - 1 \\ 3 \cdot 2.27536^2 - 1.06256 - 11 \end{bmatrix} \doteq \begin{bmatrix} 1.00505 \\ 2.01703 \end{bmatrix}$$

7. (10 pts.) Given the following table, compute the divided-difference table for the tabulated function. Also write down the Newton polynomials $P_1(x)$, $P_2(x)$ and $P_3(x)$.

k	x_k	$f(x_k)$
0	0.0	0.0
1	1.0	0.3
2	2.0	3.2
3	3.0	2.5
4	4.0	5.0

k	x_k	$f(x_k)$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$	$f[\,,\,,\,,\,,\,]$
0	0	0				
1	1	0.3	$\frac{0.3-0}{1-0}=0.3$			
2	2	3.2	$\frac{3.2-0.3}{2-1}=2.9$	$\frac{2.9-0.3}{2-0}=1.3$		
3	3	2.5	$\frac{2.5-3.2}{3-2}=-0.7$	$\frac{-0.7-2.9}{3-1}=-1.8$	$\frac{-1.8-1.3}{3-0}=-\frac{31}{30}$	
4	4	5.0	$\frac{5-2.5}{4-3}=2.5$	$\frac{2.5+0.7}{4-2}=1.6$	$\frac{1.6+1.8}{4-1}=\frac{34}{30}$	$\frac{\frac{34}{30}+\frac{31}{30}}{4-0}=\frac{13}{24}$

$p_1(X)=0.3X$

$p_2(X)=0.3X+1.3X(X-1)$

$P_3(X)=0.3X+1.3X(X-1)-\frac{31}{30}X(X-1)(X-2)$

8. (10 pts.) Using Pade' approximations find $R_{2,2}(x)$ for $f(x)=e^x$. Start with the Maclaurin expansion:

$P_2(x)=p_0+p_1x+p_2x^2$
$Q_2(x)=1+q_1x+q_2x^2$

$$f(x)=1+x+x^2/2+x^3/6+x^4/24$$

$f(x)\cdot Q_2(x)-P_2(x)=0$

$$\Rightarrow \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}\right)(1+q_1x+q_2x^2)-(p_0+p_1x+p_2x^2)\equiv 0$$

$$\Rightarrow \begin{cases} 1-p_0=0 \\ q_1+1-p_1=0 \\ q_1+q_2+\frac{1}{2}-p_2=0 \\ q_2+\frac{q_1}{2}+\frac{1}{6}=0 \\ \frac{q_2}{2}+\frac{q_1}{6}+\frac{1}{24}=0 \end{cases} \Rightarrow \begin{cases} p_0=1 \\ p_1=\frac{1}{2} \\ p_2=\frac{1}{12} \\ q_1=-\frac{1}{2} \\ q_2=\frac{1}{12} \end{cases}$$

$$R_{2,2}(x)=\frac{P_2(x)}{Q_2(x)}=\frac{1+\frac{1}{2}x+\frac{1}{12}x^2}{1-\frac{1}{2}x+\frac{1}{12}x^2}=\frac{x^2+6x+12}{x^2-6x+12}$$

Method 2: Newton interpolation

use divided difference to determine the parabola function

k	x_k	$f(x_k)$	$f(\,,\,)$	$f(\,,\,,\,)$
0	x_{k-3}	y_{k-3}		
1	x_{k-2}	y_{k-2}	$\frac{y_{k-2}-y_{k-3}}{x_{k-2}-x_{k-3}}$	
2	x_{k-1}	y_{k-1}	$\frac{y_{k-1}-y_{k-2}}{x_{k-1}-x_{k-2}}$	$\frac{\frac{y_{k-1}-y_{k-2}}{x_{k-1}-x_{k-2}}-\frac{y_{k-2}-y_{k-3}}{x_{k-2}-x_{k-3}}}{x_{k-1}-x_{k-3}}$

$$\Rightarrow P_3(x)=y_{k-3}+\frac{y_{k-2}-y_{k-3}}{x_{k-2}-x_{k-3}}(x-x_{k-3})+\frac{\frac{y_{k-1}-y_{k-2}}{x_{k-1}-x_{k-2}}-\frac{y_{k-2}-y_{k-3}}{x_{k-2}-x_{k-3}}}{x_{k-1}-x_{k-3}}(x-x_{k-3})(x-x_{k-2})$$

$$\Rightarrow P_3(x)=\left[\frac{\frac{y_{k-1}-y_{k-2}}{x_{k-1}-x_{k-2}}-\frac{y_{k-2}-y_{k-3}}{x_{k-2}-x_{k-3}}}{x_{k-1}-x_{k-3}}\right]\cdot x^2+$$

$$\left[-\frac{\left(\frac{y_{k-1}-y_{k-2}}{x_{k-1}-x_{k-2}}-\frac{y_{k-2}-y_{k-3}}{x_{k-2}-x_{k-3}}\right)(x_{k-3}+x_{k-2})}{x_{k-1}-x_{k-3}}+\frac{y_{k-2}-y_{k-3}}{x_{k-2}-x_{k-3}}\right]\cdot x+$$

$$\left[\frac{\left(\frac{y_{k-1}-y_{k-2}}{x_{k-1}-x_{k-2}}-\frac{y_{k-2}-y_{k-3}}{x_{k-2}-x_{k-3}}\right)\cdot x_{k-3}x_{k-2}}{x_{k-1}-x_{k-3}}-\frac{(y_{k-2}-y_{k-3})x_{k-3}}{x_{k-2}-x_{k-3}}+y_{k-3}\right]$$

$\Rightarrow P_3(x)=ax^2+bx+c$ $\quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ choose the root close to x_{k-1}

a, b, c are obtained finally check the convergence is the solution.

9. (15 pts.)The secant method works by approximating the function f(x) using a line at the point (x_{k-1},y_{k-1}) and (x_{k-2},y_{k-2}) and it determines x_k as the intersection of this line with the x-axis. Write an improved method to determine x_k that uses an additional third point (x_{k-3},y_{k-3}) to approximate f(x) using a parabola that passes through these three points. If the iteration equation gives more than one root, choose x_k to be the closest to the solution. Write the iteration equation for the method.

Method1. As the question states, the parabola passes $(x_{k-1},y_{k-1}),(x_{k-2},y_{k-2})(x_{k-3},y_{k-3})$

use lagrange interpolation to determine the parabola function

$$f(x)=y_{k-1}\frac{(x-x_{k-2})(x-x_{k-3})}{(x_{k-1}-x_{k-2})(x_{k-1})(x_{k-3})}+y_{k-2}\frac{(x-x_{k-1})(x-x_{k-3})}{(x_{k-2}-x_{k-1})(x_{k-2}-x_{k-3})}+y_{k-3}\frac{(x-x_{k-1})(x-x_{k-2})}{(x_{k-3}-x_{k-1})(x_{k-3}-x_{k-2})}$$

$\Rightarrow f(x)=0 \Rightarrow$ collect and simplify to get the standard quadratic function

$f(x)=ax^2+bx+c=0$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2}$, choose the root close to x_{k-1}