$$P_3(x) = \frac{.5}{\frac{\pi}{6}\cdot\left(-\frac{\pi}{6}\right)\left(-\frac{\pi}{3}\right)}(x)\left(x-\frac{\pi}{3}\right)\left(x-\frac{\pi}{2}\right)$$

$$+ \frac{.866}{\left(\frac{\pi}{3}\right)\left(\frac{\pi}{6}\right)\left(-\frac{\pi}{6}\right)}(x)\left(x-\frac{\pi}{6}\right)\left(x-\frac{\pi}{2}\right)$$

$$+ \frac{1}{\left(\frac{\pi}{2}\right)\left(\frac{\pi}{3}\right)\left(\frac{\pi}{6}\right)}(x)\left(x-\frac{\pi}{6}\right)\left(x-\frac{\pi}{3}\right)$$

$$P_3(x) = 1.74(x)(x-1.047)(x-1.5708)$$

$-.5236$ $\quad$ -1.5708

$$- 3.0164(x)\left(x-\frac{\pi}{6}\right)\left(x-\frac{\pi}{2}\right)$$

$$+ 1.1611(x)\left(x-\frac{\pi}{6}\right)\left(x-\frac{\pi}{3}\right)$$

$-.5236$ $\quad$ -1.047

Now lets evaluate $P_3(x)$ at $x = \frac{\pi}{4} = .7854$

$\sin\left(\frac{\pi}{4}\right) = .7071$ exact value

$$P_3(.7854) = 1.74(.7854)(.7854-1.0472)(.7854-1.5708)$$
$$- 3.0164(.7854)(.7854-.5236)(.7854-1.5708)$$
$$+ 1.1611(.7854)(.7854-.5236)(.7854-1.0472)$$

$$= .28078 + .4871 - .06246$$

$$P_3(.7854) = .7054$$

Exact Solution $\sin(.7854) = .7071$

Newton Polynomials

- It is an alternative to Lagrange approximation to build polynomials.
- Given $N+1$ points $(x_0, y_0), (x_1, y_1) \dots (x_N, y_N)$ we want to build a polynomial of degree N that passes through this points
- Newton Polynomials can be built incrementally, that is, the work done for $P_{N-1}(x)$ can be used to build $P_N(x)$.

Assume:

$$P_1(x) = a_0 + a_1(x - x_0)$$

$$P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

$$P_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$$\vdots$$

$$P_N(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots \dots a_N(x - x_0)(x - x_1)\dots(x - x_{N-1})$$

How to compute $a_0, a_1, \ldots a_N$

Assume we want to build $P_1(X)$ with the points (X_0, Y_0) and (X_1, Y_1)

From the definition $P_1(X) = a_0 + a_1(X - X_0)$

$$Y_0 = P_1(X_0) = a_0 + a_1 \cancel{(X_0 - X_0)}^{\,0}$$

then $\underline{\underline{a_0 = Y_0}}$ ①

$$P_1(X_1) = Y_1 = a_0 + a_1(X_1 - X_0)$$

From ①

$$Y_1 = Y_0 + a_1(X_1 - X_0)$$

$$Y_1 - Y_0 = a_1(X_1 - X_0)$$

$$a_1 = \frac{Y_1 - Y_0}{X_1 - X_0} \quad ②$$

Now if we want to build $P_2(x)$ with an additional point (x_2, y_2)

$$P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

$$P_2(x_2) = y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

From 1 and 2

$$y_2 = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

we want to obtain a_2

$$y_2 - y_0 - \frac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0) = a_2(x_2 - x_0)(x_2 - x_1)$$

$$a_2 = \frac{1}{(x_2 - x_0)(x_2 - x_1)}\left[y_2 - y_0 - \frac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0)\right]$$

$$a_2 = \frac{1}{x_2 - x_1}\left[\frac{y_2 - y_0}{(x_2 - x_0)} - \frac{y_1 - y_0}{x_1 - x_0}\right]$$

$$a_2 = \frac{1}{x_2 - x_1}\left[\frac{(y_2 - y_0)(x_1 - x_0) - (y_1 - y_0)(x_2 - x_0)}{(x_2 - x_0)(x_1 - x_0)}\right]$$

$$+ y_1x_1 - y_1x_1$$

$$a_2 = \frac{y_2x_1 - y_0x_1 - y_2x_0 + \cancel{y_0x_0} - y_1x_2 + y_0x_2 + y_1x_0 - \cancel{y_0x_0}}{(x_2 - x_1)(x_2 - x_0)(x_1 - x_0)}$$

Now we add $y_1x_1 - y_1x_1 = 0$ to the numerator

The we factor

$$a_2 = \frac{y_2(x_1 - x_0) - y_1(x_1 - x_0) - y_1(x_2 - x_1) + y_0(x_2 - x_1)}{(x_2 - x_1)(x_2 - x_0)(x_1 - x_0)}$$

$$a_2 = \frac{(y_2 - y_1)(x_1 - x_0) - (y_1 - y_0)(x_2 - x_1)}{(x_2 - x_1)(x_2 - x_0)(x_1 - x_0)}$$

$$a_2 = \frac{1}{x_2 - x_0}\left[\frac{(y_2 - y_1)\cancel{(x_1 - x_0)}}{(x_2 - x_1)\cancel{(x_1 - x_0)}} - \frac{(y_1 - y_0)\cancel{(x_2 - x_1)}}{\cancel{(x_2 - x_1)}(x_1 - x_0)}\right]$$

so we get:

$$a_2 = \frac{1}{x_2 - x_0}\left[\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}\right] \quad (3)$$

That is "the slope of the slope" in (x_2, y_2), (x_1, y_1) and (x_1, y_1), (x_0, y_0)

The divided differences of a function $f(x)$ are defined as follows

$$f[x_k] = y_k$$

$$f[x_{k-1}, x_k] = \frac{f[x_k] - f[x_{k-1}]}{x_k - x_{k-1}}$$

$$f[x_{k-2}, x_{k-1}, x_k] = \frac{f[x_{k-1}, x_k] - f[x_{k-2}, x_{k-1}]}{x_k - x_{k-2}}$$

$$\vdots$$

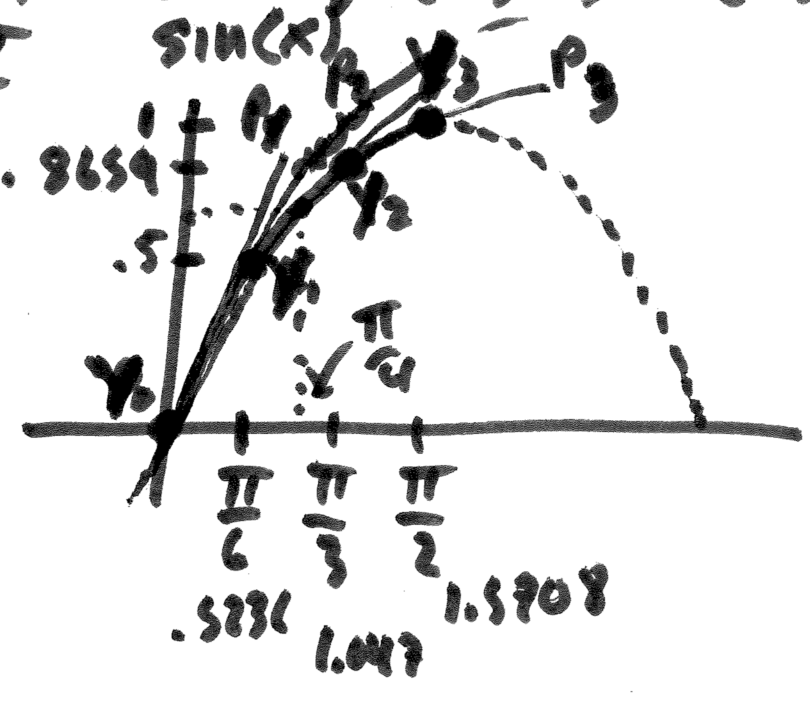
$$f[x_{k-j}, x_{k-j+1} \ldots x_k] = \frac{f[x_{k-j+1} \ldots x_k] - f[x_{k-j} \ldots x_{k-1}]}{x_k - x_{k-j}}$$

where $a_k = f[x_0, x_1, \ldots x_k]$

we express the divided difference in a table

Example.

Build polinomials of degree 1, 2 and 3 to approximate $f(x) = \sin(x)$ in the interval $0, \frac{\pi}{2}$



Divided Difference Table

x_k	$f[x_k]$	$f[x_{k-1},x_k]$	$f[x_{k-2},x_{k-1},x_k]$	$f[x_{k-3},x_{k-2},x_{k-1},x_k]$
$x_0=0$	$y_0=0$ ← a_0			
		$\frac{.5-0}{.5236-0} = .9549$ ← a_1		
$x_1=\frac{\pi}{6}=.5236$	$y_1=.5$			
			$\frac{.699-.9549}{1.047-0} = -.2444$ ← a_2	
		$\frac{.8659-.5}{1.047-.5236} = .699$		
$x_2=\frac{\pi}{3}=2\frac{\pi}{6}=1.047$	$y_2=.8659$			$-.2444$
				$\frac{-.4230-(-.2444)}{1.5708-0} = -.1137$ ← a_3
		$\frac{1-.8659}{1.5708-1.047} = .256$		
			$\frac{.256-.699}{1.5708-.5236} = -.4230$	
$x_3=\frac{\pi}{2}=1.5708$	$y_3=1$			

We have that

$$P_1(x) = a_0 + a_1(x - x_0) \to P_2(x) = P_1(x) + a_2(x-x_0)(x-x_1)$$

110

$$P_2(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$$

$$P_3(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$$

So we get

Approximate $\sin(x)$ with $P_1(x)$, $P_2(x)$ and $P_3(x)$.

$$P_1(x) = \overset{a_0}{0} + \overset{a_1}{.9549}(x - \overset{x_0}{0}) = .9549x$$

$$P_2(x) = P_1(x) + a_2(x-x_0)(x-x_1).$$

$$= .9549x + (-.2444)(x-0)(x-.5236)$$

$$= .9549x - .2444x(x-.5236)$$

$$P_3(x) = P_2(x) + a_3(x-x_0)(x-x_1)(x-x_2)$$

$$= .9549x - .2444x(x-.5236) + (-.1137)(x-0)(x-.5236)(x-1.047$$

Evaluating $P_3(x)$ at $x = \frac{\pi}{4} = .7854$

Exact Solution

$$\sin\left(\frac{\pi}{4}\right) = \underline{\underline{.7071}}$$

$$P_3(.7854) = .9549(.7854) - .2444(.7854)(.7854 - .5236) - .1137(.7854)(.7854 - .5236)(.7854 - 1.047)$$

$$= \underline{\underline{.70584}}$$

Padé Approximations

They are used to approximate functions with a ratio of two polinomials:

$$R_{N,M}(x) = \frac{P_N(x)}{Q_M(x)} \quad \text{for } a \le x \le b$$

where

① $P_N(x) = P_0 + P_1 x + P_2 x^2 \dots \quad P_N (x)^N$

② $Q_M(x) = 1 + q_1 x + q_2 x^2 + \dots Q_M (x)^M$

Assume that $f(x)$ has the following Maclaurin expansion (Maclaurin expansion = taylor expansion when $x_0 = 0$)

③ $f(x) = a_0 + a_1 x + a_2 x^2 + \dots \quad a_k x^k + \dots$

also we have that

$$f(x) = \frac{P_N(x)}{Q_M(x)} \Rightarrow f(x)\, Q_M(x) - P_N(x) = 0 \quad ④$$

Sustituting 1 and 2 and 3 into 4

$$f(X)\,Q_M(X) - P_N(X) = 0$$

$$\underbrace{(a_0 + a_1 X + a_2 X^2 + \ldots a_K X^K \ldots)}_{f(X)}(1 + q_1 X + q_2 X^2 + \ldots q_M (X)^M)$$

$$- (P_0 + P_1 X + P_2 X^2 + \ldots P_N (X)^N) = 0$$

Performing factorization and multiplication

$$(a_0 - P_0) + X(a_1 + a_0 q_1 - P_1) + x^2(a_2 + q_1 a_1 + q_2 a_0 - P_2)$$

$$+ x^3(\ldots\ldots \quad) \ldots\ldots \quad = 0$$

Since we want the left side to be ϕ regardless of the value of X, we make every coefficient equal to ϕ.

$\cancel{a_0 - p_0 = 0}$

$$a_0 - p_0 = 0$$

$$q_1 a_0 + a_1 - p_1 = 0$$

$$q_2 a_0 + q_1 a_1 + a_2 - p_2 = 0$$

$$\vdots$$

$$q_M a_{N-M} + q_{M-1} a_{N-M+1} \cdots a_N - p_N = 0$$

$$q_M a_{N-M+1} + q_{M-1} a_{N-M+2} + \cdots a_{N+1} = 0$$

$$\vdots \qquad \vdots$$

$$q_M a_N + q_{M-1} a_{N+1} \cdots a_{N+M} = 0$$

$N+M+1$ equations

We get $N+M+1$ equations to determine $q_1, q_2 \cdots q_M$ and $p_0, p_1 \cdots p_N$

Example

Find the Padé approximation $R_{2,2}(x)$ for $f(x) = \ln(1+x)/x$

Start with the Mclaurin expansion

① $f(x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \dots$

Sol.

$$R_{N,M}(x) = \frac{P_N(x)}{Q_M(x)}$$

$P_2(x)$ ①

$$R_{2,2}(x) = \frac{P_2(x)}{Q_2(x)} = \frac{p_0 + p_1 x + p_2 x^2}{1 + q_1 x + q_2 x^2}$$

$Q_2(x)$ ②

we want

$$f(x) = \frac{P_N(x)}{Q_N(x)}$$

③ $f(x)\, Q_N(x) - P_N(x) = 0$

$2 \qquad 2$

substituting 0, 1, 2 into 3

$$\left(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}\cdots\right)\left(1+q_1x+q_2x^2\right)-\left(p_0+p_1x+p_2x^2\right)=0$$

we need 5 equations to determine

q_1, q_2, p_0, p_1, p_2.

We obtain then by multiplying

and factoring x.

$$1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}+q_1x-\frac{q_1}{2}x^2+\frac{q_1}{3}x^3-\frac{q_1}{4}x^4+\frac{q_1}{5}x^5\cdots$$

$$+q_2x^2-\frac{q_2}{2}x^3+\frac{q_2}{3}x^4-\frac{q_2}{4}x^5+\frac{q_2}{5}x^6\cdots$$

$$-p_0-p_1x-p_2x^2=0$$

Factoring X we get

$$(1-p_0)+x\left(-\frac{1}{2}+q_1-p_1\right)+x^2\left(\frac{1}{3}-\frac{q_1}{2}+q_2-p_2\right)$$

$$+x^3\left(-\frac{1}{4}+\frac{q_1}{3}-\frac{q_2}{2}\right)+x^4\left(\frac{1}{5}-\frac{q_1}{4}+\frac{q_2}{3}\right)\ldots\ldots=0$$

So we get the equations

1. $1-p_0=0$
2. $-\frac{1}{2}+q_1-p_1=0$
3. $\frac{1}{3}-\frac{q_1}{2}+q_2-p_2=0$
4. $-\frac{1}{4}+\frac{q_1}{3}-\frac{q_2}{2}=0$
5. $\frac{1}{5}-\frac{q_1}{4}+\frac{q_2}{3}=0$

q_1 ✓
p_2 ✓
q_2 ✓
p_1 ✓
p_0

Multiplying 4 by 3 and 5 by 4 and adding

$$-\frac{3}{4} + \cancel{q_1} - \frac{3}{2} q_2 = 0$$

$$+ \quad \frac{4}{5} - \cancel{q_1} + \frac{4}{3} q_2 = 0$$

$$-\frac{3}{4} + \frac{4}{5} - \frac{3}{2} q_2 + \frac{4}{3} q_2 = 0$$

$$\frac{-15+16}{20} + q_2\left(-\frac{3}{2} + \frac{4}{3}\right) = 0$$

$$\frac{1}{20} + q_2 \frac{-9+8}{6} = 0$$

$$\frac{1}{20} - \frac{q_2}{6} = 0$$

$$\frac{q_2}{6} = \frac{1}{20}$$

$$\boxed{q_2 = \frac{6}{20} = \frac{3}{10}}$$

From 4 sustituting q_2

$$q_1 = \left(\frac{q_2}{2} + \frac{1}{4}\right)3$$

$$= \left(\frac{3}{10}\frac{1}{2} + \frac{1}{4}\right)3$$

$$= \left(\frac{3}{20} + \frac{1}{4}\right)3 = \left(\frac{3+5}{20}\right)3$$

$$= \frac{8}{20}\cdot 3 = \frac{24}{20} = \frac{6}{5}$$

$$\boxed{q_1 = \frac{6}{5}}$$

From 3:

$$p_2 = \frac{1}{3} - \frac{q_1}{2} + q_2$$

$$p_2 = \frac{1}{3} - \frac{6}{5}\frac{1}{2} + \frac{3}{10}$$

$$= \frac{1}{3} - \frac{3}{5} + \frac{3}{10} = \frac{10-18+9}{30} = \frac{1}{30}$$

$$\boxed{p_2 = \frac{1}{30}}$$

From ②

$$p_1 = -\frac{1}{2} + q_1$$

$$= -\frac{1}{2} + \frac{6}{5} = \frac{-5+12}{10} = \frac{7}{10}$$

$$\boxed{p_1 = \frac{7}{10}}$$

From ① $\boxed{p_0 = 1}$

So we have

$$P_{2,2}(x) = \frac{p_0 + p_1 x + p_2 x^2}{1 + q_1 x + q_2 x^2}$$

$$= \frac{1 + \frac{7}{10}x + \frac{1}{30}x^2}{1 + \frac{6}{5}x + \frac{3}{10}x^2}$$

For $x=1$

$$f(x)=\frac{\ln(1+x)}{x}$$

$$f(1)=\frac{\ln(1+1)}{1}=\underline{\underline{.6931}}$$ exact value

Using Pade approximation

$$f(x)\approx P_{2,2}(x)=\frac{1+\frac{7}{10}(1)+\frac{1}{30}(1)^2}{1+\frac{6}{5}(1)+\frac{3}{10}(1)^2}$$

$$=\underline{\underline{.6933}}$$

If we want only to approximate $\ln(1+x)$ then we do

$$\frac{\ln(1+x)}{x}\approx\frac{1+\frac{7}{10}x+\frac{1}{30}x^2}{1+\frac{6}{5}x+\frac{3}{10}x^2}$$

$$\ln(1+x) \approx x \, \frac{1 + \frac{7}{10}x + \frac{1}{30}x^2}{1 + \frac{6}{5}x + \frac{3}{10}x^2} \quad (121)$$

$$\approx \frac{x + \frac{7}{10}x^2 + \frac{1}{30}x^3}{1 + \frac{6}{5}x + \frac{3}{10}x^2} \; \frac{(30)}{(30)}$$

$$\ln(1+x) \approx \frac{30x + 21x^2 + x^3}{30 + 36x + 9x^2}$$

Let $y = 1+x$

$x = y - 1$

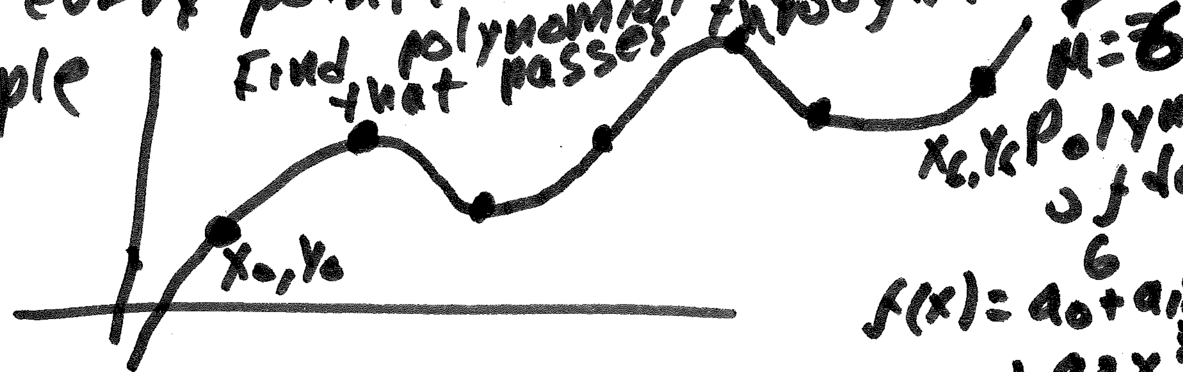
$$\boxed{\ln(y) \approx \frac{30(y-1) + 21(y-1)^2 + (y-1)^3}{30 + 36(y-1) + 9(y-1)^2}}$$

Curve Fitting

- Given a set of points $(x_1, y_1), (x_2, y_2) \ldots (x_N, y_N)$ find the curve that best fits these points

Polynomial Approximation - Given $M+1$ points find polynomial of degree M that passes through every point.

Example



$(x_0, y_0) \ldots (x_6, y_6)$

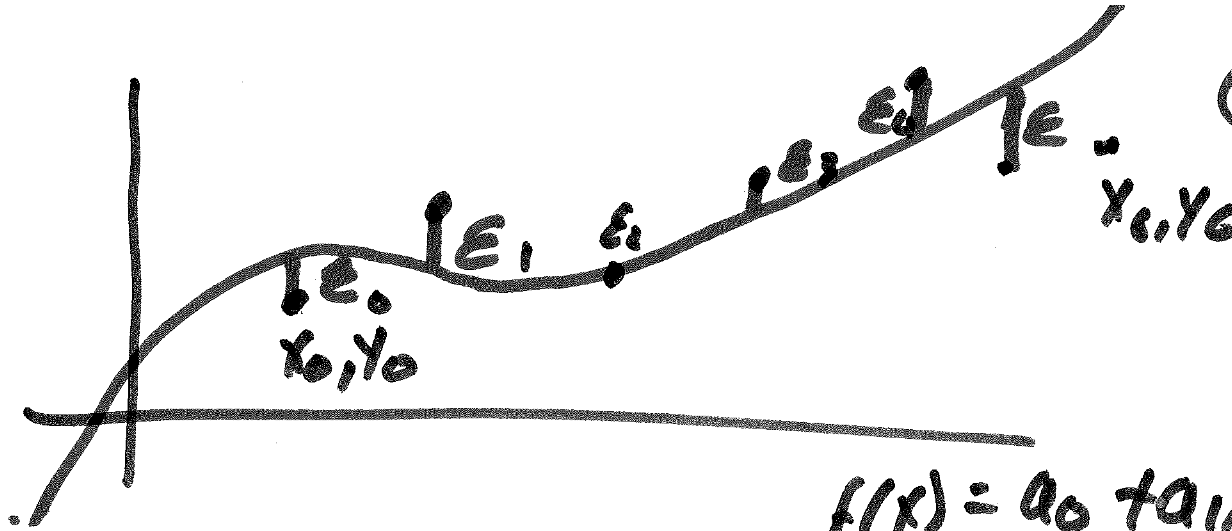
$M = 6$

Polynomial of degree 6

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6$

Curve Fitting - The curve does not necessarily touches the points but passes close to them.

Example: Find the cubic polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ that passes close to points $(x_0, y_0) \ldots (x_6, y_6)$ that minimizes error.



$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Since we have more points that we need to build the cubic polynomial, we find a_0, a_1, a_2, a_3 such that the error $e = \sum_{i=0}^{n} |E_i|$ is minimized.

Least Squares line

Assume we have recorded values (x_1, y_1), (x_2, y_2) ... (x_N, y_N) and we want to find the line $y = f(x) = Ax + B$ that best fits the recorded values.

We have that

sample

$$f(x_k) = y_k + e_k$$

approximation   error.

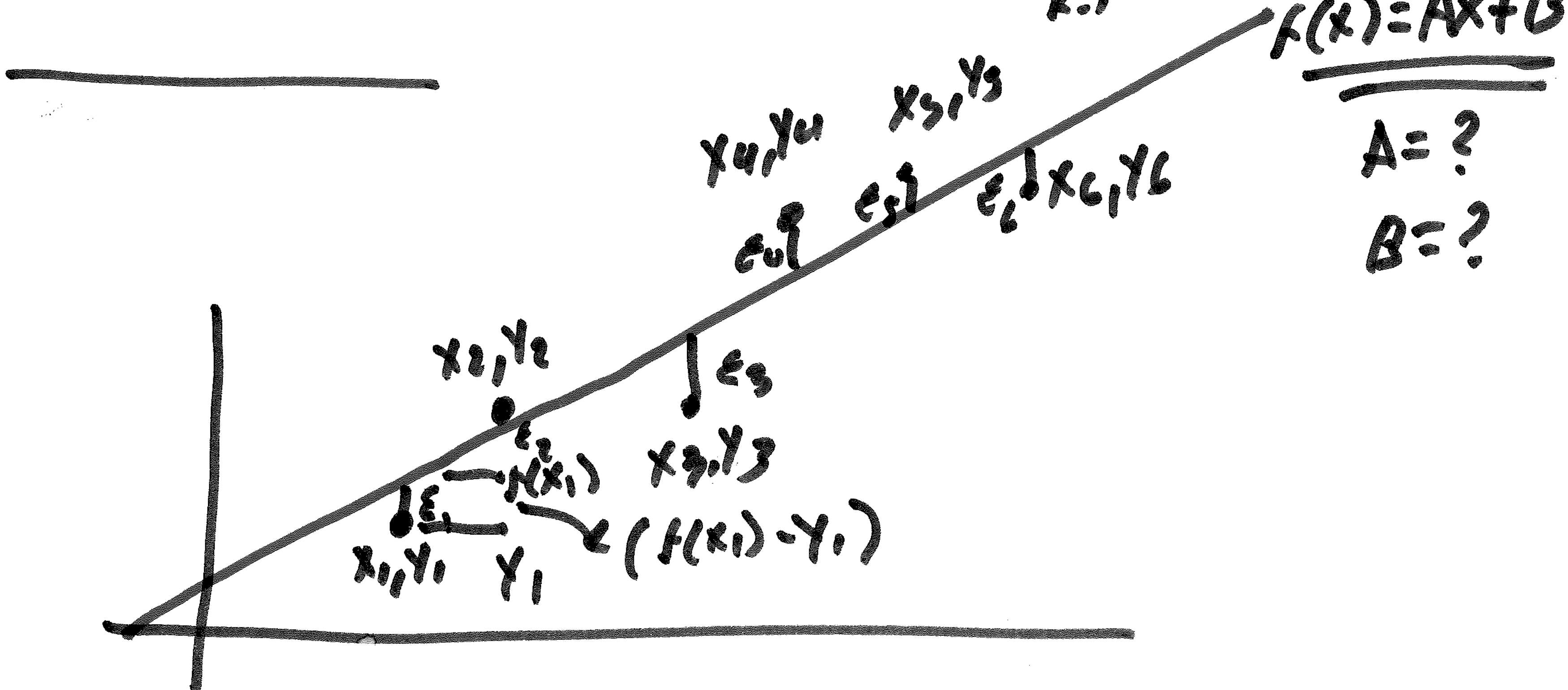
There are several ways to measure the error that can be uses to tell us how far $f(x)$ is from the data.

Maximum Error $\quad E_\infty(f) = \max\limits_{1 \le k \le N} \{|f(x_k) - y_k|\}$

Average Error $\quad E_1(f) = \frac{1}{N} \sum_{k=1}^{N} |f(x_k) - y_k|$

~~Root Mean~~

Root Mean Square Error $E_2(f) = \frac{1}{N}\sum_{k=1}^{N}(f(x_k) - y_k)^2$ 12(



We choose "Root Mean Square Error" since it is easier to use to obtain a derivative to obtain the minimum error.

Given points $(x_1, y_1)(x_2, y_2) \ldots (x_N, y_N)$ (127)
the least squares line $y = f(x) = Ax + B$
is the line that minimizes the square error $E_2(f)$

$$E(A,B) = \sum_{k=1}^{N} (\overbrace{Ax_k + B - y_k}^{f(x)\,-\,y_k})^2$$

To obtain the minimum we obtain

$$\frac{\partial E(A,B)}{\partial A} = \sum_{k=1}^{N} 2(Ax_k + B - y_k)x_k$$

$$= 2\sum_{k=1}^{N} (Ax_k^2 + Bx_k - y_k x_k) \quad (1)$$

$$\frac{\partial E(A,B)}{\partial B} = \sum_{k=1}^{N} 2(Ax_k + B - y_k)(1) \quad (2)$$

To obtain the minimum error, we need to set $\frac{\partial E(A,B)}{\partial A} = \emptyset$

and $\frac{\partial E(A,B)}{\partial B} = 0$ and obtain A, B

$$\frac{\partial E(A,B)}{\partial A} = 0 = \cancel{2}\sum_{K=1}^{N}(A X_K^2 + B X_K - Y_K X_K) = 0$$

$$\boxed{\sum_{K=1}^{N} A X_K^2 + \sum_{K=1}^{N} B X_K - \sum_{K=1}^{N} Y_K X_K = 0}$$

Least Squares eq 1

Also

$$\frac{\partial E(A,B)}{\partial B} = 0 = \sum_{K=1}^{N}\cancel{2}(A X_K + B - Y_K)$$

$$A\sum_{K=1}^{N} X_K + B\sum_{K=1}^{N}\cancel{(1)}^{N} - \sum_{K=1}^{N} Y_K = 0$$

Least squares eq 2 $\quad A\sum_{K=1}^{N} X_K + BN - \sum_{K=1}^{N} Y_K = 0$

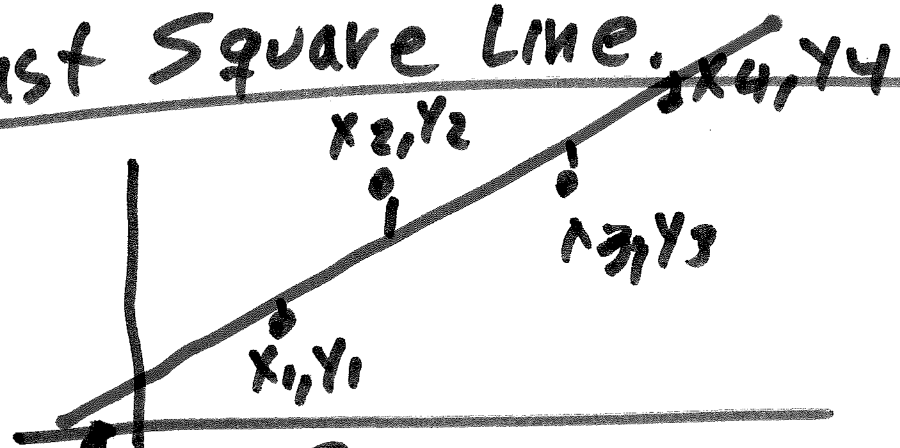
So we get two linear equations to obtain A, B

$$\text{①}\; A\sum_{k=1}^{N} X_k^2 + B\sum_{k=1}^{N} X_k - \sum_{k=1}^{N} Y_k X_k = 0$$

and

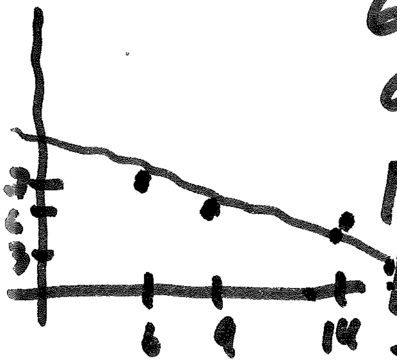
$$\text{②}\; A\sum_{k=1}^{N} X_k + B\,N - \sum_{k=1}^{N} Y_k = 0$$

Least Square Line.



$y = \underline{AX + B}$

Example: Assume the following data:



X_K	Y_K	X_K^2	$X_K Y_K$
6	7	36	42
9	6	81	54
14	3	196	42
17	1	289	17
21	0	441	0
$\sum X_K = 67$	$\sum Y_K = 17$	$\sum X_K^2 = 1043$	$\sum X_K Y_K = 155$

From ① $\quad A(1043) + B(67) - 155 = 0 \quad$ ③

From ② $\quad A(67) + B(5) - 17 = 0 \quad$ ④

Multiplying ③ by -5 and ④ by 67 and adding both

$$A(1043)(-5) + \cancel{B(67)(-5)} - 155(-5) = 0$$

$$+ \quad A(67)(67) + \cancel{B(5)(67)} - 17(67) = 0$$

$$A\left(1043(-5) + (67)^2\right) - \left(155(-5) + 17(67)\right) = 0$$

$$A(-726) - 364 = 0$$

$$A = \frac{364}{-726} = -.5013$$

From ③ $\cancel{(-.5013)(67) + B}$

$$(-.5013)(1043) + B(67) - 155 = 0$$

$$B = \frac{155 + (.5013)(1043)}{67}$$

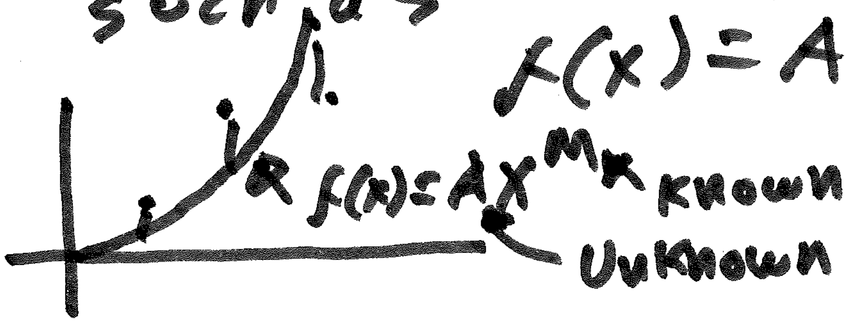
$$B = 10.11$$

So the line that minimizes the error will be.

$$y = Ax + B$$

$$\boxed{y = (-.5013)x + 10.11}$$

Sometimes we would like to fit the data points to other functions such as



$$f(x) = A X^M$$

where M is a known constant and A is unknown.

We also want to minimize the square error

$$E(A) = \sum_{K=1}^{N} (A X_K^M - Y_K)^2$$

$$\frac{\partial E}{\partial A} = \sum_{K=1}^{N} \cancel{2}(A X_K^M - Y_K) X_K^M = 0$$

$$\sum_{K=1}^{N} (A X_K^M X_K^M - Y_K X_K^M) = 0$$

$$\sum_{K=1}^{N} (A X_K^{2M} - Y_K X_K^M) = 0$$

$$A \sum_{K=1}^{N} X_K^{2M} - \sum_{K=1}^{N} Y_K X_K^{M} = 0$$

$$\boxed{A = \frac{\sum_{K=1}^{N} Y_K X_K^{M}}{\sum_{K=1}^{N} X_K^{2M}}}$$

Least Squares for $f(x) = AX^M$

Example

we want to fit data to $Y = Ax^3$ minimizing the error find A.

$Y = AX^3$

$A = ?$

X_K	Y_K	$Y_K X_K^3$	X_K^6
2.0	5.9	47.2	64
2.3	8.3	100.986	148.04
2.6	10.7	188.06	308.92
2.9	13.7	334.13	594.82
3.2	17.0	557.06	1,073.74
		1,227.44	2189.52

$M = 3$

$$\sum_{K=1}^{N} Y_K X_K^M = \sum Y_K X_K^3 \qquad \sum X_K^{2M} = \sum X_K^6$$

$$A = \frac{\sum_{K=1}^{N} Y_K X_K^M}{\sum_{K=1}^{N} X_K^{2M}} = \frac{\sum_{K=1}^{N} Y_K X_K^3}{\sum_{K=1}^{N} X_K^6} = \frac{1,227.44}{2189.52} = .5606$$

$$\boxed{Y = .5606\, X^3}$$

CS314 Midterm Review

- Floating Point Binary Representation
- Propagation of Errors
- Solution of Nonlinear Equations
  - Fixed Point Theorem
  - Types of convergence/divergence
    + Monotone Convergence
    + Oscilating Convergence
    + Monotone Divergence
    + Oscilating Divergence
  - Bisection Method
  - False Position Method
    + Horizontal Convergence $|f(x)| < \epsilon$
    + Vertical Convergence $|x_k - x_{k-1}| < \epsilon$
    + Well conditioned / ill conditioned roots.

- Newton-Raphson
  - Similarities of Newton-Raphson and Taylor exponsion.
  - Order of convergence
- Secant Method

- Solution of Linear Systems

$$\mathbb{A}\mathbb{X}=\mathbb{B}$$

- Properties of Vectors
- Vector Algebra
- Matrices
- Properties of Matrices
- Special Matrices
  + Zero Matrix
  + Identity Matrix
  + Matrix Multiplilation
  + Inverse of a Matrix

- Upper Triangular System of Equations
- Backward Sustitution
- Elemantary Transformations
- Gauss Elimination ← One system of equations
  - + Pivot
  - + Choosing Pivot
- LU Factorization (Triangular Factorization)
  - + For multiple systems of equations that have the same $\mathbb{A}$

$$
\begin{aligned}
\mathbb{A}\,\mathbb{X}_1 &= \mathbb{B}_1 \\
\mathbb{A}\,\mathbb{X}_2 &= \mathbb{B}_2 \\
&\vdots \\
\mathbb{A}\,\mathbb{X}_M &= \mathbb{B}_M
\end{aligned}
$$

- Gauss Elimination vs. Triangular Factorization

- Iterative Methods for Linear Equations
  - Gauss-Seidel
  - Jakobi
  - Convergence and Strictly Diagonal Dominant Matrices.
- Solution of systems of non-linear equations.
  - Newton Method.

Interpolation and Polinomial Approximation
- Taylor Approximation
- Horners method for evaluating polinomials.
- Lagrange Approximation
- Newton Polinomials
  - Divided Differences
- Pade Approximation of functions using a ratio of polinomials.

to study:
- Class Notes
- Book
- Homeworks
- I will post HW3 today to be due on Wed.

Exam Th and Fr:

Curve Fitting

Non-linear least squares method for
$Y = Ce^{AX}$

Suppose we have points $(X_1, Y_1) \ldots (X_N, Y_N)$ and we want to fit the points to an exponential curve

$$f(X) = Y = Ce^{AX}$$

We want to find A, C that minimizes the square error.

$$E(A,C) = \sum_{K=1}^{N} \left(f(X_K) - Y_K\right)^2$$

$$= \sum_{K=1}^{N} \left(Ce^{AX_K} - Y_K\right)^2$$

$$\frac{\partial E}{\partial A} = \sum_{k=1}^{N} 2\left(Ce^{Ax_k} - y_k\right) C x_k e^{Ax_k} = 0$$

$$\sum_{k=1}^{N} \left(\cancel{2}C^2 x_k e^{Ax_k} e^{Ax_k} - \cancel{2} y_k C x_k e^{Ax_k}\right) = 0$$

$$\sum_{k=1}^{N} \left(C^2 x_k e^{2Ax_k} - y_k C x_k e^{Ax_k}\right) = 0$$

$$\sum_{k=1}^{N} C^{\cancel{2}} x_k e^{2Ax_k} - \sum_{k=1}^{N} y_k \cancel{C} x_k e^{Ax_k} = 0$$

$$(1) \quad C \sum_{k=1}^{N} x_k e^{2Ax_k} - \sum_{k=1}^{N} y_k x_k e^{Ax_k} = 0$$

$$\frac{\partial E}{\partial C} = \sum_{k=1}^{N} \cancel{2}\left(Ce^{Ax_k} - y_k\right)e^{Ax_k} = 0$$

see ~~see~~.

$$(2) \quad C\sum_{k=1}^{N} e^{2Ax_k} - \sum_{k=1}^{N} y_k e^{Ax_k} = 0$$

So with (1) and (2) we obtain a system of non-linear equations that can be used to solve A and C using **Newton's method** for non-linear equations.

Transformations for Data Linearization

It is possible to fit curves such as

$y = C e^{Ax}$, $y = A \ln(x) + B$ and $y = A/x + B$

by transforming these equations into a linear curve $y = Ax + B$

Example:

$$y = \underline{C}\, e^{Ax} \Rightarrow \ln(y) = \ln(C e^{Ax})$$

$$\ln(y) = \ln(c) + \ln(e^{Ax})$$

$$\underbrace{\ln(y)}_{z} = \underbrace{\ln(c)}_{B} + Ax$$

$z = \ln(y)$

$z_k = \ln(y_k)$

$B = \ln(c)$

$$\underline{z = Ax + B}$$

In this way we obtain least squares for $z = Ax + B$ instead of $y = Ce^{Ax}$

Assume data points 146

(0, 1.5), (1, 2.5), (2, 3.5), (3, 5.0), (4, 7.5)

we want to fit to $y = Ce^{Ax}$

X_K	Y_K	$Z_K = \ln(Y_K)$	X_K^2	$X_K Z_K$
0	1.5	.4055	0	0
1	2.5	.9163	1	.9163
2	3.5	1.2528	4	2.5056
3	5.0	1.6094	9	4.8282
4	7.5	2.0149	16	8.0596
$\sum_1^N X_K$ 10		$\sum_1^N Z_K$ 6.1984	$\sum_1^N X_K^2$ 30	$\sum_1^N X_K Z_K$ 16.3097

$N = 5$

Using least squares line

$$0 = A \sum_{K=1}^{N} X_K^2 + B \sum_{K=1}^{N} X_K - \sum_{K=1}^{N} X_K Z_K$$

$$0 = A \sum_{K=1}^{N} X_K + BN - \sum_{K=1}^{N} Z_K$$

so we have:

$$0 = A(30) + B(10) - 16.3097 \quad ①$$

and

$$0 = A(10) + B(5) - 6.1989 \quad ② \ (-2)$$

Multiply ② by -2 and add to ①

$$0 = A(\underbrace{30 - 20}_{10}) + \cancel{B(10-10)}^{0} - 16.3097 + (2)6.1989$$

$$A = \frac{16.3097 - 2(6.1989)}{10} = .3912$$

$$\boxed{A = .3912}$$

$$B = \frac{16.3097 - A(30)}{10}$$

$$= \frac{16.3097 - (.3912)(30)}{10}$$

$$\boxed{B = .4574}$$

So we have

$$Z = Ax + B$$

But we want $\quad y = Ce^{Ax}$

$$B = \ln(c) \implies c = e^{B}$$

$$c = e^{B} = e^{.4574} = 1.5799$$

So we get that

$$\boxed{y = (1.5799)\, e^{.3912x}}$$

minimizes the error using linearization and least squares line.

If we use the no-linear least squares method for $y = Ce^{Ax}$ it gives $y = 1.61089 45 e^{.38357x}$

Other sostitutions that can be used for linearization are:

$$Y = \frac{A}{X} + B \quad \rightarrow \quad y = A\Big(\underbrace{\frac{1}{x}}_{z}\Big) + B$$

$$Y = A z + B$$

or

$$Y = C x^{A} \quad \rightarrow \quad \underbrace{\ln(Y)}_{W} = \underbrace{\ln(c)}_{B} + A \underbrace{\ln(x)}_{z}$$

$$W = A z + B$$

Linear least Squares

Suppose that we have N data points $(x_1, y_1), (x_2, y_2) \dots (x_N, y_N)$ and a set of M linear dependent functions $f_j(x)$

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + \dots c_M f_M(x)$$

We want to obtain $c_1, c_2, \dots c_M$ that minimizes the square error to approximate $(x_1, y_1), (x_2, y_2) \dots (x_N, y_N)$.

$$E(c_1, c_2 \dots c_M) = \sum_{k=1}^{N} (f(x_k) - y_k)^2$$

$$= \sum_{k=1}^{N} (c_1 f_1(x_k) + c_2 f_2(x_k) \dots c_M f_M(x_k) - y_k)^2$$

$$\frac{\partial E}{\partial c_i} = \sum_{k=1}^{N} 2\left(c_1 f_1(x_k) + c_2 f_2(x_k) \cdots c_M f_M(x_k) - y_k\right) f_i(x_k) = 0$$

$$\cancel{2} \sum_{k=1}^{N} \left[\left[\sum_{j=1}^{M} c_j f_j(x_k)\right] - y_k\right] f_i(x_k) = 0$$

$$\sum_{k=1}^{N} \left[\left[\sum_{j=1}^{M} c_j f_j(x_k)\right] f_i(x_k) - y_k f_i(x_k)\right] = 0$$

$$\sum_{k=1}^{N} \left[\left[\sum_{j=1}^{M} c_j f_j(x_k)\right] f_i(x_k)\right] - \sum_{k=1}^{N} y_k f_i(x_k) = 0$$

$$\boxed{\sum_{k=1}^{N} \left[\left[\sum_{j=1}^{M} c_j f_j(x_k)\right] f_i(x_k)\right] = \sum_{k=1}^{N} y_k f_i(x_k) \quad \text{for } i = 1 \ldots M}$$

This gives us M equations to solve M variables

When the linear least squares is adapted to a polynomial we have:

$$f(x) = c_1 + c_2 x + c_3 x^2 + \dots \quad c_{M+1} x^M$$

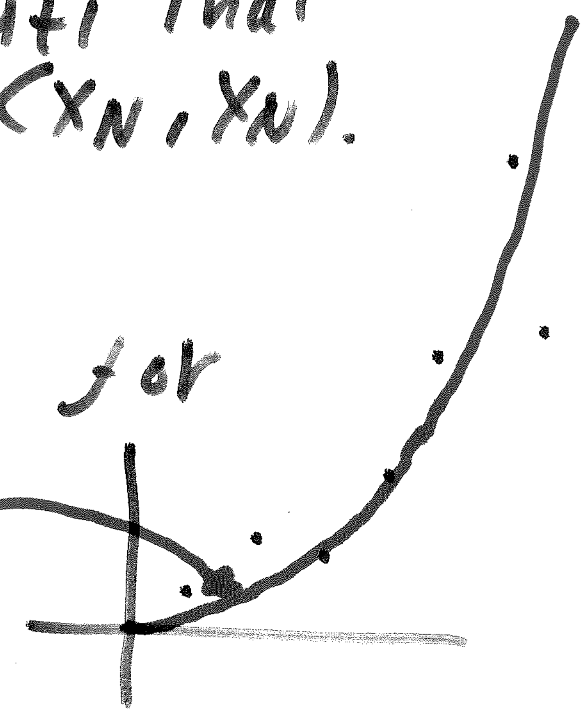
We want to find $c_1, c_2 \dots c_{M+1}$ that best fits $(x_1, y_1), (x_2, y_2) \dots (x_N, y_N)$.

<u>Example</u>

We want to find A, B, C for

$$f(x) = Ax^2 + Bx + C$$

that fits $(x_1, y_1), (x_2, y_2) \dots (x_N, y_N)$



We first find the error function

$$E(A,B,C) = \sum_{K=1}^{N} \left(AX_K^2 + BX_K + C - Y_K\right)^2$$

$$\frac{\partial E}{\partial A} = \sum_{K=1}^{N} \cancel{2}\left(AX_K^2 + BX_K + C - Y_K\right)X_K^2 = 0$$

① $$A\sum_{K=1}^{N} X_K^4 + B\sum_{K=1}^{N} X_K^3 + C\sum_{K=1}^{N} X_K^2 = \sum_{K=1}^{N} Y_K X_K^2$$

$$\frac{\partial E}{\partial B} = \sum_{K=1}^{N} \cancel{2}\left(AX_K^2 + BX_K + C - Y_K\right)X_K = 0$$

② $$A\sum_{K=1}^{N} X_K^3 + B\sum_{K=1}^{N} X_K^2 + C\sum_{K=1}^{N} X_K = \sum_{K=1}^{N} Y_K X_K$$

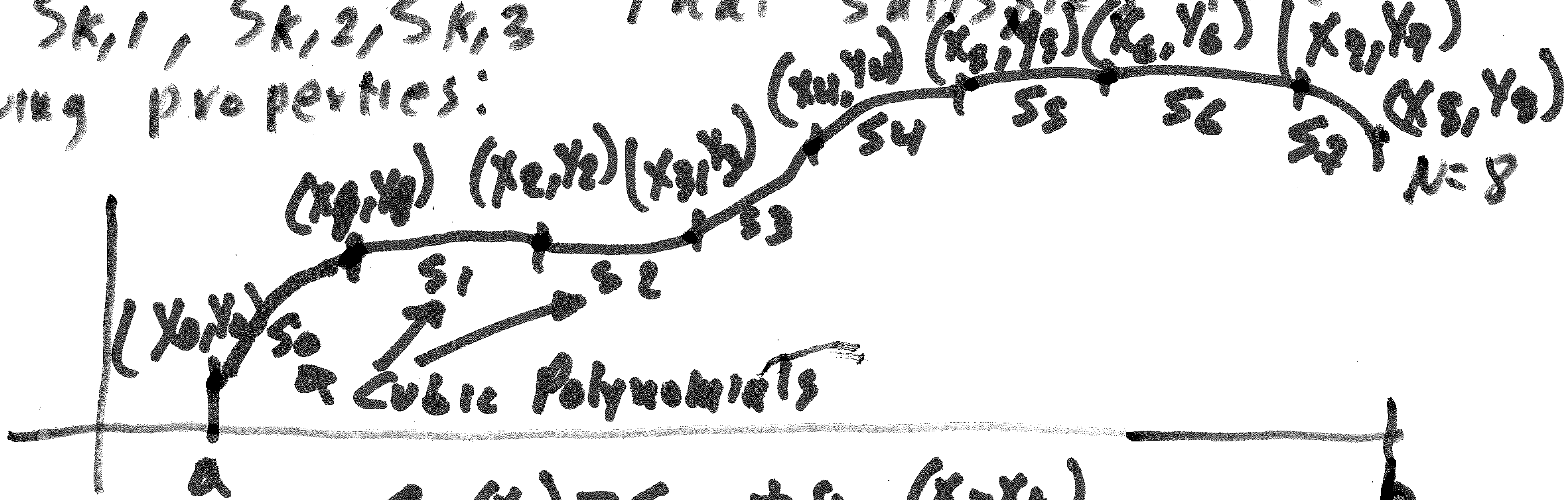
$$\frac{\partial E}{\partial C} = \sum_{k=1}^{N} \cancel{2}(A x_k^2 + B x_k + C - y_k)(1) = 0$$

$$(3) \quad A \sum_{k=1}^{N} x_k^2 + B \sum_{k=1}^{N} x_k + CN = \sum_{k=1}^{N} y_k$$

Solving (1), (2), and (3) we will find A, B and C the minimize the error to fit $(x_1, y_1) \dots (x_N, y_N)$ into $y = Ax^2 + Bx + C$.

Piecewise Cubic Splines

We want to fit $N+1$ data points $(x_0, y_0), (x_1, y_1) \ldots (x_N, y_N)$ where $a = x_0 < x_1 < x_2 \ldots < x_N = b$ into <u>N cubic polynomials</u> $S_k(x)$ with coefficients $S_{k,0}, S_{k,1}, S_{k,2}, S_{k,3}$ that satisfies the following properties:



$$S_k(x) = S_{k,0} + S_{k,1}(x - x_k) + S_{k,2}(x - x_k)^2 + S_{k,3}(x - x_k)^3$$

Spline Properties

I. $S(x) = S_k(x) = S_{k,0} + S_{k,1}(x-x_k) + S_{k,2}(x-x_k)^2 + S_{k,3}(x-x_k)^3$

for $x \in [x_k, x_{k+1}]$ and $k = 0, 1, \ldots N-1$

→ A cubic polynomial between each pair of points

II $S_k(x_k) = y_k$ for $k = 0, 1, N$

The polynomial passes throug each point.

III $S_k(x_{k+1}) = S_{k+1}(x_{k+1})$ for $k = 0, 1, \ldots N-2$

The polynomials are connected to each other

IV $S'_k(x_{k+1}) = S'_{k+1}(x_{k+1})$ For $k = 0, 1, \ldots N-2$

Continuity in the first derivative

V $S''_k(x_{k+1}) = S''_{k+1}(x_{k+1})$ For $k = 0, 1 \ldots N-2$

Continuity in the second derivative.

For each polynomial $S_K(x)$ there are four coefficients to be determined $S_{K0}, S_{K,1}, S_{K,2}, S_{K,3}$ (4 degrees of freedom)

So since there are N polynomials, there are

<u>$4N$</u> coefficients to be determined.

- From II and with $N+1$ data points we have $N+1$ constraints.
- From III, IV and V each supplies $N+1$ constraints for a total of $3(N-1)$ constraints

adding them

$$\begin{array}{r} N+1 \\ +\ 3(N-1) \\ \hline N+1+3N-3 \end{array}$$

$= 4N-2$ constraints

Since we need $4N$ constraints and we only have $4N-2$, there are two additional degrees of freedom that we can control.

We call these degrees of freedom the "end point" constraints and involve $S'(x)$ and $S''(x)$ at x_0 and x_N and will be discussed later.

Building the Splines

Since $S_k(x)$ is cubic, the second derivative is a line. We can obtain the equation of the line using "Lagrange Approximation".

$$S_k''(x) = S''(x_k)\,\frac{x - x_{k+1}}{x_k - x_{k+1}} + S''(x_{k+1})\,\frac{x - x_k}{x_{k+1} - x_k}$$

so $S_k''(x) = S''(x_k)$ when $x = x_k$

$S_k''(x) = S''(x_{k+1})$ when $x = x_{k+1}$

Use $m_k = s''(x_k)$, $m_{k+1} = s''(x_{k+1})$

and $h_k = x_{k+1} - x_k$

so we get

$$S_k''(x) = m_k \frac{x_{k+1} - x}{h_k} + m_{k+1} \frac{x - x_k}{h_k}$$

Integrating $s_k''(x)$ twice

$$S_k(x) = \int_{\cancel{x_k}}^{\cancel{x_{k+1}}} \int_{\cancel{x_k}}^{\cancel{x_{k+1}}} S_k''(x)\,dx\,dx = \int_{\cancel{x_k}}^{\cancel{x_{k+1}}} \int_{\cancel{x_k}}^{\cancel{x_{k+1}}} m_k \frac{(x_{k+1} - x)}{h_k} + m_{k+1} \frac{(x - x_k)}{h_k}\,dx\,dx$$

$$= \int_{\cancel{x_k}}^{\cancel{x_{k+1}}} -\frac{m_k}{2h_k}(x_{k+1} - x)^2 + \frac{m_{k+1}}{2h_k}(x - x_k)^2 + c_1\,dx$$

$$= \frac{m_k}{6h_k}(x_{k+1} - x)^3 + \frac{m_{k+1}}{6h_k}(x - x_k)^3 + c_1 x + c_2$$

We can express $c_1 x + c_2$ as

$$c_1 x + c_2 = p_k (x_{k+1} - x) + q_k (x - x_k)$$

being p_k and q_k two constants.

So we get

$$① \quad S_k(x) = \frac{m_k}{6h_k}(x_{k+1}-x)^3 + \frac{m_{k+1}}{6h_k}(x-x_k)^3 + p_k(x_{k+1}-x) + q_k(x-x_k)$$

Sustituting $x = x_k$, $S(x_k) = y_k$ in ①

$$S_k(x_k) = y_k = \frac{m_k}{6h_k}(\overbrace{x_{k+1}-x_k}^{h_k})^3 + \cancel{\frac{m_{k+1}}{6h_k}(x_k - x_k)^3} \to 0 + p_k(\underbrace{x_{k+1}-x_k}_{h_k}) + \cancel{q_k(x_k - x_k)} \to 0$$

$$y_k = \frac{m_k}{6\cancel{h_k}} h_k^{\cancel{3}\,2} + p_k h_k$$

$$y_k = \frac{m_k}{6} h_k^2 + p_k h_k$$

So we get

$$p_k = \left(y_k - \frac{m_k}{6} h_k^2 \right) \frac{1}{h_k}$$

$$2 \quad p_k = \frac{y_k}{h_k} - \frac{m_k h_k}{6}$$

Also sustituting $x = x_{k+1}$ and $s(x_{k+1}) = y_{k+1}$ in 1 we get

$$S_k(x_{k+1}) = y_{k+1} = \frac{m_k}{6h_k}\cancel{(x_{k+1} - x_{k+1})}^{3} + \frac{m_{k+1}}{6h_k}(\overbrace{x_{k+1} - x_k}^{h_k})^3$$

$$+ p_k \cancel{(x_{k+1} - x_{k+1})} + q_k(\overbrace{x_{k+1} - x_k}^{h_k})$$

$$y_{k+1} = \frac{m_{k+1}}{6 \cancel{h_k}} (h_k^{\cancel{3}\,2}) + q_k(h_k)$$

$$y_{k+1} = \frac{m_{k+1}}{6} h_k^2 + q_k h_k$$

so we get

$$q_k = \left(y_{k+1} - \frac{m_{k+1} h_k^2}{6} \right) \frac{1}{h_k}$$

$$(3) \quad q_k = \frac{y_{k+1}}{h_k} - \frac{m_{k+1} h_k}{6}$$

Sustituting (2) and (3) into (1)

$$S_k(x) = \frac{m_k}{6h_k}(x_{k+1}-x)^3 + \frac{m_{k+1}}{6h_k}(x-x_k)^3$$
$$+ \left(\frac{y_k}{h_k} - \frac{m_k h_k}{6}\right)(x_{k+1}-x)$$
$$+ \left(\frac{y_{k+1}}{h_k} - \frac{m_{k+1} h_k}{6}\right)(x-x_k)$$

So ~~t~~ the spline is taking form. Only the terms m_k and m_{k+1} are unknown.

To obtain these values we differentiate $S_k(x)$.

$$(4) \quad S_k'(x) = \frac{3m_k}{6h_k}(x_{k+1}-x)^2(-1) + \frac{3m_{k+1}}{6h_k}(x-x_k)^2$$
$$+ \left(\frac{y_k}{h_k} - \frac{m_k h_k}{6}\right)(-1) + \left(\frac{y_{k+1}}{h_k} - \frac{m_{k+1} h_k}{6}\right)$$

Evaluating $S_k'(x)$ at $x = x_k$

$$S_k'(x_k) = \frac{3m_k}{6h_k}(\underbrace{x_{k+1} - x_k}_{h_k})^2(-1) + \frac{3m_{k+1}}{6h_k}\cancel{(x_k - x_k)^2}^{\,0}$$

$$+\left(\frac{y_k}{h_k} - \frac{m_k h_k}{6}\right)(-1) + \left(\frac{y_{k+1}}{h_k} - \frac{m_{k+1}h_k}{6}\right)$$

$$= -\frac{3m_k}{6\cancel{h_k}}h_k^{\cancel{2}} - \frac{y_k}{h_k} + \frac{m_k h_k}{6} + \frac{y_{k+1}}{h_k} - \frac{m_{k+1}h_k}{6}$$

$-\frac{1}{2}$

$$= m_k h_k\left(\underbrace{-\frac{1}{2} + \frac{1}{6}}_{-1/3}\right) - \frac{m_{k+1}h_k}{6} + \frac{y_{k+1} - y_k}{h_k}$$

d_k

Let $d_k = \dfrac{y_{k+1} - y_k}{h_k}$

So we get

(5) $$S_k'(x_k) = \frac{-m_k h_k}{3} - \frac{m_{k+1}h_k}{6} + d_k$$

Sustituting k by $k-1$ in 4

$$S_{k-1}'(x) = \frac{1}{2}\frac{m_{k-1}}{h_{k-1}}(x_k - x)^2(-1) + \frac{1}{2}\frac{m_k}{h_{k-1}}(x - x_{k-1})^2$$
$$+ \left(\frac{y_{k-1}}{h_{k-1}} - \frac{m_{k-1}h_{k-1}}{6}\right)(-1)$$
$$+ \left(\frac{y_k}{h_{k-1}} - \frac{m_k h_{k-1}}{6}\right)$$

Evaluating $S_{k-1}'(x)$ at $x = x_k$

$$S_{k-1}'(x_k) = \frac{1}{2}\frac{m_{k-1}}{h_{k-1}}\cancel{(x_k - x_k)^2}(-1) + \frac{1}{2}\frac{m_k}{h_{k-1}}(\overbrace{x_k - x_{k-1}}^{h_{k-1}})^2$$
$$+ \left(\frac{y_{k-1}}{h_{k-1}} - \frac{m_{k-1}h_{k-1}}{6}\right)(-1)$$
$$+ \left(\frac{y_k}{h_{k-1}} - \frac{m_k h_{k-1}}{6}\right)$$

$$= \frac{1}{2}\frac{m_k}{\cancel{h_{k-1}}} h_{k-1}^{\cancel{2}} - \frac{y_{k-1}}{h_{k-1}} + \frac{m_{k-1}h_{k-1}}{6}$$
$$+ \frac{y_k}{h_{k-1}} - \frac{m_k h_{k-1}}{6}$$

(164)

$$S_{k-1}'(x_k) = m_k h_{k-1}\left(\frac{1}{2} - \frac{1}{6}\right) + \frac{m_{k-1}h_{k-1}}{6}$$

$\frac{1}{3}$

$$+ \frac{y_k - y_{k-1}}{h_{k-1}}$$

Let $d_{k-1} = \frac{y_k - y_{k-1}}{h_{k-1}}$

(6) $$S_{k-1}'(x_k) = \frac{m_k h_{k-1}}{3} + \frac{m_{k-1} h_{k-1}}{6} + d_{k-1}$$

From $\overline{\underline{IV}}$ $s_k'(x_k) = S_{k-1}'(x_k)$

and (5) and (6)

$$s_k'(x_k) = S_{k-1}'(x_k)$$

$$-\frac{m_k h_k}{3} - \frac{m_{k+1} h_k}{6} + d_k = \frac{m_k h_{k-1}}{3} + \frac{m_{k-1} h_{k-1}}{6} + d_{k-1}$$

multiplying both sides by 6 we get

$$-2m_k h_k - m_{k+1} h_k + 6d_k = 2m_k h_{k-1} + m_{k-1} h_{k-1} + 6d_{k-1}$$

$$6(d_k - d_{k-1}) = m_{k-1} h_{k-1} + 2m_k(h_{k-1} + h_k) + m_{k+1} h_k$$

Let $U_k = 6(d_k - d_{k-1})$

we have

(7) $m_{k-1} h_{k-1} + m_k 2(h_{k-1} + h_k) + m_{k+1} h_k = U_k$

for $k = 1, 2 \ldots N-1$

this will give us $N-1$ equations

but we need to find $m_0, m_1 \ldots m_N$

that are $N+2$ constants.

- The other two equations will be given by the "end point constraints".

One of the "end point" constraints is to assume that $m_0 = S''(x_0)$ and $m_N = S''(x_N)$

with this assumption we have that from (7)

for $k=1$ and (7)

$$m_0 h_0 + m_1 (2)(h_0 + h_1) + m_2 h_1 = U_1$$

$$\Downarrow$$

(8) $$m_1 (2)(h_0 + h_1) + m_2 h_1 = U_1 - m_0 h_0$$

for $k = N-1$ and 7

$$m_{N-2} h_{N-2} + m_{N-1} (2)(h_{N-2} + h_{N-1}) + m_N h_{N-1} = U_{N-1}$$

$$\Downarrow$$

9 $$m_{N-2} h_{N-2} + m_{N-1} (2)(h_{N-2} + h_{N-1}) = U_{N-1} - m_N h_{N-1}$$

From 7, 8, and 9 we get a system of $N-1$ equations with $m_1 \ldots m_{N-1}$ unknowns.

$$\begin{bmatrix} b_1 & c_1 & & & \phi \\ a_2 & b_2 & c_2 & & \\ & a_3 & b_3 & c_3 & \\ & \phi & & & \ddots \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_{N-1} \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{N-1} \end{bmatrix}$$

this system is strictly diagonal dominant.

After the coefficients $m_1, m_2 \ldots m_{N-1}$ are computed, we want to build the coefficients $S_{k,0}$, $S_{k,1}$, $S_{k,2}$, $S_{k,3}$ for every spline from $k=0 \ldots N-1$.

for $$S_k(x) = S_{k,0} + S_{k,1}(x-x_k) + S_{k,2}(x-x_k)^2 + S_{k,3}(x-x_k)^3$$

these ~~values~~ coefficients can be obtained with the equations:

$$S_{k,0} = y_k, \quad S_{k,1} = d_k - \frac{h_k(2m_k + m_{k+1})}{6}$$

$$S_{k,2} = \frac{m_k}{2} \qquad S_{k,3} = \frac{m_{k+1} - m_k}{6h_k}$$

these equations come from the following.

we have that

$$S_k(x) = S_{k,0} + S_{k,1}(x-x_k) + S_{k,2}(x-x_k)^2 + S_{k,3}(x-x_k)^3$$

substituting $x = x_k$ we have

$$S_k(x_k) = y_k = S_{k,0} + S_{k,1}\overset{0}{\cancel{(x_k-x_k)}} + S_{k,2}\overset{0}{\cancel{(x_k-x_k)}}^2 + S_{k,3}\overset{0}{\cancel{(x_k-x_k)}}^3$$

so we get that

$$S_{k,0} = y_k$$

If we derivate $S(x)$ we get

$$\text{(A)} \quad S_k'(x) = S_{k,1} + 2S_{k,2}(x-x_k) + 3S_{k,3}(x-x_k)^2$$

If we derivate again.

$$S_k''(x) = 2S_{k,2} + 6S_{k,3}(x - x_k)$$

Evaluating at $x = x_k$

$$S_k''(x_k) = 2S_{k,2} + 6S_{k,3}\underbrace{\cancel{(x_k - x_k)}}_{0}$$

$$S_k''(x_k) = 2S_{k,2}$$

From the definition of $m_k = S''(x_k)$ we get

$$m_k = 2S_{k,2} \Rightarrow S_{k,2} = \frac{m_k}{2}$$

If we make $S_k'(x)$ in (A) with $x = x_k$ we get.

$$S_k'(x_k) = S_{k,1} + 2S_{k,2}\underbrace{\cancel{(x_k - x_k)}}_{0} + 3S_{k,3}\underbrace{\cancel{(x_k - x_k)}}_{0}^{2}$$

$$S_k'(x_k) = S_{k,1}$$

From 5

$$S_k'(x_k) = -\frac{m_k h_k}{3} - \frac{m_{k+1} h_k}{6} + d_k$$

so we get

$$S_{k,1} = S_k'(x_k) = -\frac{m_k h_k}{3} - \frac{m_{k+1} h_k}{6} + d_k$$

$$S_{k,1} = d_k - \frac{h_k(2m_k + m_{k+1})}{6}$$

Making $x = x_{k+1}$ in $S''(x)$ $\quad \underbrace{h_k}$

$$S_k''(x_{k+1}) = 2\,S_{k,2} + 6\,S_{k,3}(x_{k+1} - x_k)$$

Also since $m_{k+1} = S_k''(x_{k+1})$ also $S_{k,2} = \frac{m_k}{2}$

$$m_{k+1} = 2\,S_{k,2} + 6\,S_{k,3}\,h_k$$

$$m_{k+1} = \cancel{2}\frac{m_k}{\cancel{2}} + 6\, s_{k,3}\, h_k$$

$$s_{k,3} = \frac{m_{k+1} - m_k}{6 h_k}$$

End point constraints

We need two more constraints to eliminate m_0 and m_N. These two constraints are called "end point" constraints.

End Point Constraints:

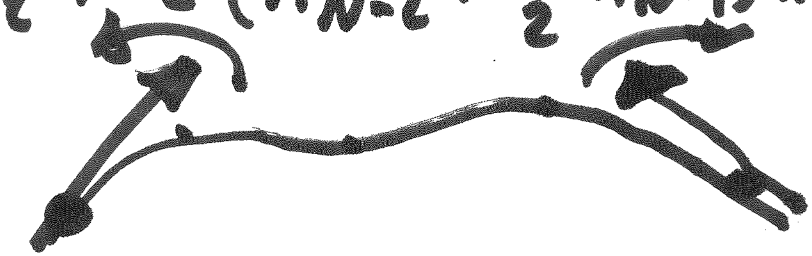
1. Clamped Spline. The first derivative at the ends are given. $S'(a)$ and $S'(b)$.

$$\left(\frac{3}{2}h_0+2h_1\right)m_1+h_1m_2=u_1-3\left(d_0-S'(x_0)\right)$$

$$h_{k-1}m_{k-1}+2(h_{k-1}+h_k)m_k+h_km_{k+1}=u_k$$

$$k=2,3\ldots N-2$$

$$h_{N-2}m_{N-2}+2\left(h_{N-2}+\frac{3}{2}h_{N-1}\right)m_{N-1}=u_{N-1}-3\left(S'(x_N)-d_{N-1}\right)$$



2. Natural Spline.

The second derivatives at the ends are given:

$$S''(a) = 0 \qquad S''(b) = 0$$

$$2(h_0 + h_1)m_1 + h_1 m_2 = U_1$$

$$h_{k-1} m_{k-1} + 2(h_{k-1} + h_k)m_k + h_k m_{k+1} = U_k$$

$k = 2, 3, .. N-2$

$$h_{N-2} m_{N-2} + 2(h_{N-2} + h_{N-1}) m_{N-1} = U_{N-1}$$

3. Extrapolated Spline. (S_0 is an extension of S_1)

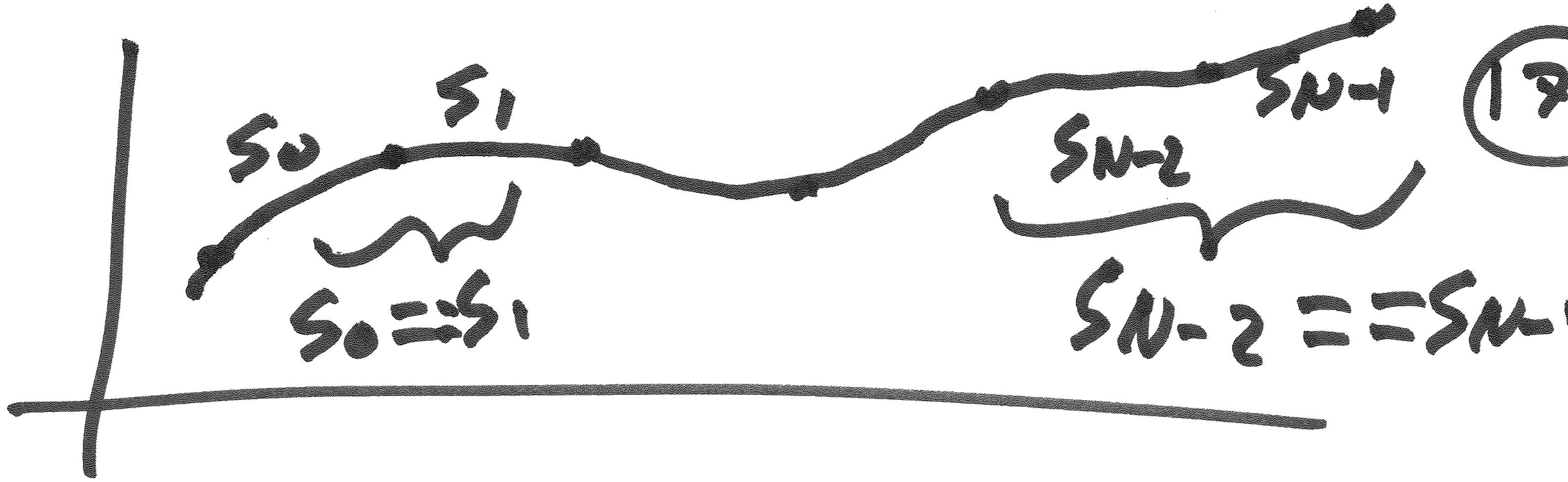
S_0 and S_1 are the same spline

S_{N-2} and S_{N-1} are the same spline (S_{N-1} is an extension of S_{N-2})

$$\left(3h_0 + 2h_1 + \frac{h_0^2}{h_1}\right)m_1 + \left(h_1 - \frac{h_0^2}{h_1}\right)m_2 = U_1$$

$$h_{k-1} m_{k-1} + 2(h_{k-1} + h_k)m_k + h_k m_{k+1} = U_k$$

$$\left(h_{N-2} - \frac{h_{N-1}^2}{h_{N-2}}\right)m_{N-2} + \left(2h_{N-2} + 3h_{N-1} + \frac{h_{N-1}^2}{h_{N-2}}\right)m_{N-1} = U_{N-1}$$



4. Parabolic Terminated Spline

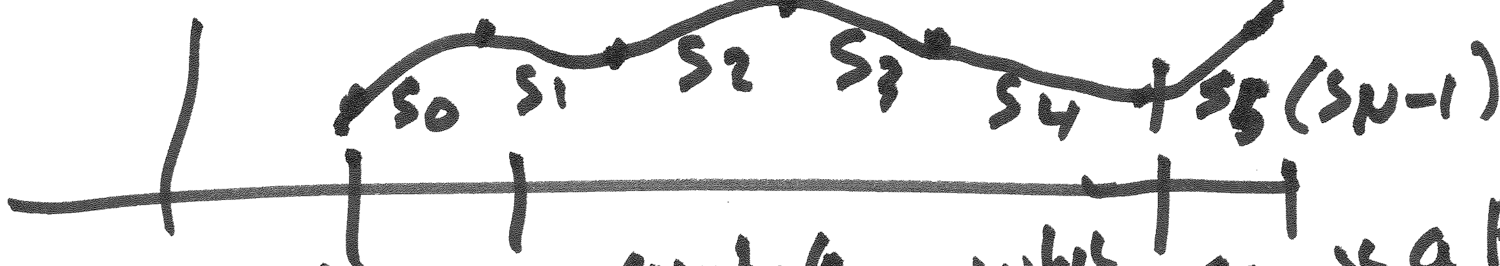
S_0 and S_{N-1} are quadratic instead of cubic

$(3h_0 + 2h_1)m_1 + h_1 m_1 = u_1$

$h_{k-1} m_{k-1} + 2(h_{k-1} + h_k)m_k + h_k m_{k+1} = u_k$

$k = 2, 3 \ldots N-2$

$h_{N-2} m_{N-2} + (2h_{N-2} + 3h_{N-1})m_{N-1} = u_{N-1}$



S_0 is a parabola instead of cubic

S_{N-1} is a parabola instead of cubic

5. End-Point ~~and~~ Curvature adjusted spline

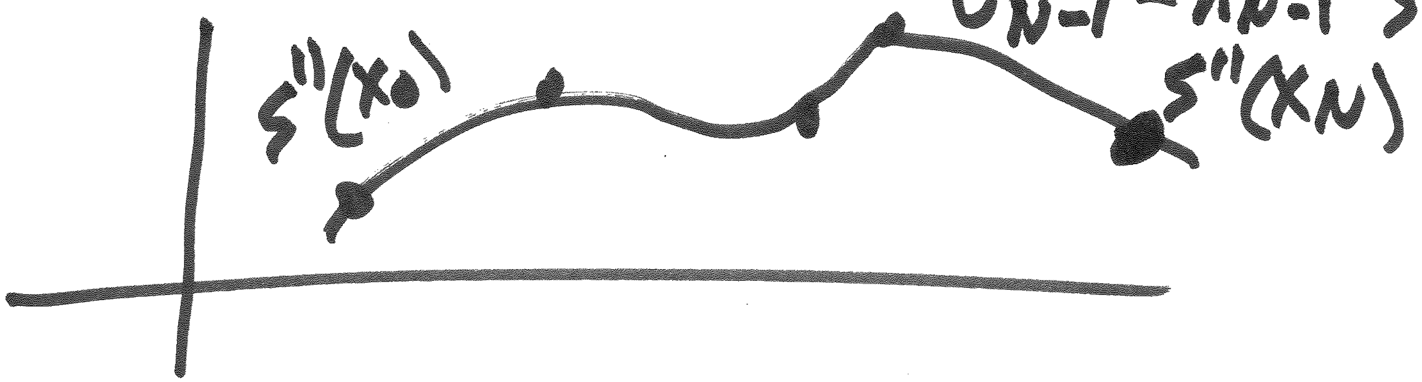
$s''(a)$ and $s''(b)$ are given

$$2(h_0 + h_1) m_1 + h_1 m_2 = u_1 - h_0 s''(x_0)$$

$$h_{k-1} m_{k-1} + 2(h_{k-1} + h_k) m_k + h_k m_{k+1} = u_k$$

$$k = 2, 3 \ldots N-2$$

$$h_{N-2} m_{N-2} + 2(h_{N-2} + h_{N-1}) m_{N-1} = u_{N-1} - h_{N-1} s''(x_N)$$

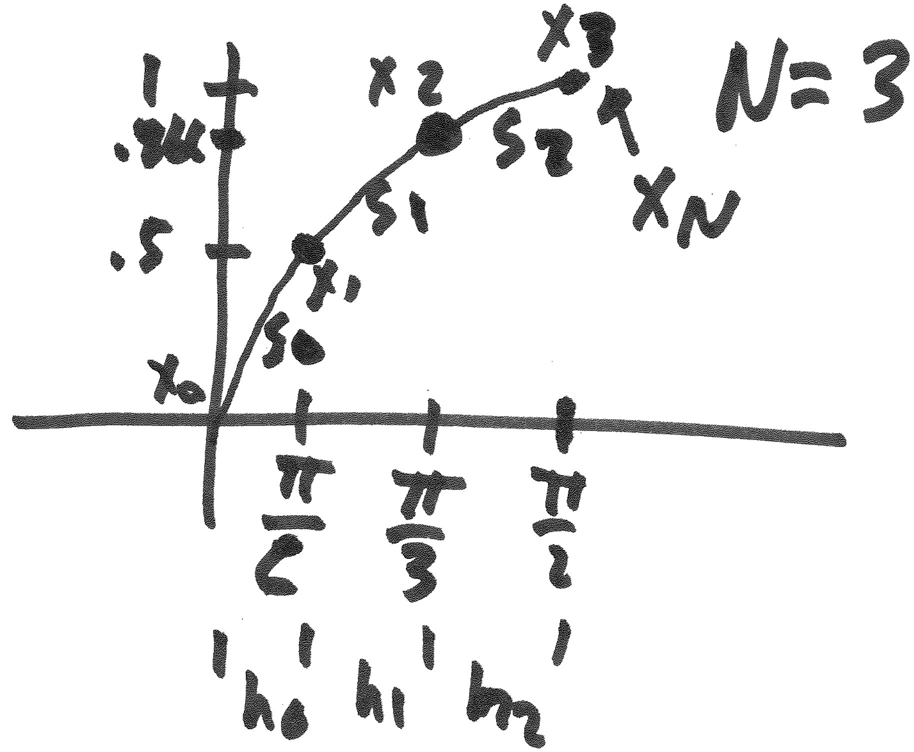


Example

$\to m_0 = 0$
$m_3 = 0$

Find the natural spline that approximates $y = \sin(x)$ in the interval $0 \to \frac{\pi}{2}$ with 4 points equally spaced.

	X	Y
0	0	0
1	$\frac{\pi}{6} \to .5236$	.5
2	$\frac{\pi}{3} \to 1.0472$	.8660
3	$\frac{\pi}{2}$	1



$h_0 = x_1 - x_0 = \frac{\pi}{6} - 0 = \frac{\pi}{6} = .5236$

$h_1 = h_2 = \frac{\pi}{6} = .5236$

—

$d_0 = \frac{y_1 - y_0}{h_0} = \frac{.5 - 0}{\frac{\pi}{6}} = .9549$

$d_1 = \frac{y_2 - y_1}{h_1} = \frac{.8660 - .5}{\pi/6} = .6990$

$d_2 = \frac{y_3 - y_2}{h_2} = \frac{1 - .8660}{\pi/6} = .2559$

—

$u_1 = 6(d_1 - d_0) = 6(.6990 - .9549) = -1.5354$

$u_2 = 6(d_2 - d_1) = 6(.2559 - .6990) = -2.6586$

From the equations of the natural spline we have:

ⓐ $2(h_0+h_1)m_1 + h_1 m_2 = U_1$

ⓑ $h_{k-1} m_{k-1} + 2(h_{k-1}+h_k) m_k + h_k m_{k+1} = U_k$
$k = 2, 3 .. N-2$

ⓒ $h_{N-2} m_{N-2} + 2(h_{N-2}+h_{N-1}) m_{N-1} = U_{N-1}$

From ⓐ

$$2\left(\frac{\pi}{6}+\frac{\pi}{6}\right)m_1 + \frac{\pi}{6} m_2 = -1.5354$$

ⓓ $2.0944 m_1 + .5236 m_2 = -1.5354$

From ⓑ there are no equations since $N = 3$ and the equations go $k = 2, 3 .. N-2$ where $N-2 = 3-2 = 1$ so $k = 2 .. 1$

no equations

From c

$$h_{N-2} m_{N-2} + 2(h_{N-2} + h_{N-1}) m_{N-1} = U_{N-1}$$

$\Rightarrow N = 3$

so we get $h_1 m_1 + 2(h_1 + h_2) m_2 = U_2$

$$\frac{\pi}{6} m_1 + 2\left(\frac{\pi}{6} + \frac{\pi}{6}\right) m_2 = -2.6586$$

e $\quad .5236 m_1 + 2.0944 m_2 = -2.6586$

So from d and e we get the two equations to determine m_1 and m_2

$$2.0944 m_1 + .5236 m_2 = -1.5354$$
$$.5236 m_1 + 2.0944 m_2 = -2.6586$$

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We use Gauss Elimination.

We use extended matrix

$$\begin{array}{c} A \\ B \end{array}\begin{bmatrix} 2.0944 & .5236 & -1.5354 \\ .5236 & 2.0944 & -2.6586 \end{bmatrix}\left(\frac{-.5236}{2.0944}\right)$$

$$A\left(\frac{-.5236}{2.0944}\right)+B \rightarrow \begin{bmatrix} 2.0944 & .5236 & -1.5354 \\ 0 & 1.9635 & -2.2748 \end{bmatrix}$$

back sustitution

$$m_2 = \frac{-2.2748}{1.9635} = -1.1585$$

$$m_1 = \frac{-1.5354 - .5236(-1.1585)}{2.0944} = -.4435$$

From m_2, m_1 we obtain the coefficients for the spline

$$S_k(x) = S_{k,0} + S_{k,1}(x - x_k) + S_{k,2}(x - x_k)^2 + S_{k,3}(x - x_k)^3$$

$$k = 0, 1, \ldots N-1$$

$$S_{k,0} = y_k$$

$$S_{k,1} = d_k - \frac{h_k(2m_k + m_{k+1})}{6}$$

$$S_{k,2} = \frac{m_k}{2}$$

$$S_{k,3} = \frac{m_{k+1} - m_k}{6h_k}$$

For $k = 0$

From natural spline.

$M_0 = S''(0) = 0$

$M_N = M_3 = S''(\frac{\pi}{2}) = 0$

$S_{0,0} = y_0 = 0$

$S_{0,1} = d_0 - \dfrac{h_0(2M_0 + M_1)}{6}$

$= .9549 - \dfrac{.5236(\cancel{2(0)}^{\,0} + (-.4435))}{6}$

$S_{0,1} = .9936$

$S_{0,2} = \dfrac{M_0}{2} = \dfrac{0}{2} = 0$

$S_{0,3} = \dfrac{M_1 - M_0}{6h_0} = \dfrac{-.4435 - 0}{6(.5236)} = -.1412$

So we have

$S_0(x) = .9936(x-0) + 0(x-0)^2 + (-.1412)(x-0)^3$

$\boxed{S_0(x) = .9936x - 1.412x^3}$

At $\frac{\pi}{6}$ $S_0(x) = .5202$

For $K=1$

$$s_{1,0} = y_1 = .5$$

$$s_{1,1} = d_1 - \frac{h_1(2m_1 + m_2)}{6}$$

$$= .6990 - \frac{.5236(2(-.4435) + (-1.1585))}{6}$$

$$= .8775$$

$$s_{1,2} = \frac{m_1}{2} = \frac{-.4435}{2} = -.2218$$

$$s_{1,3} = \frac{m_2 - m_1}{6h_1} = \frac{-1.1585 - (-.4435)}{6(.5236)} = -.2276$$

$$\boxed{S_1(x) = .5 + .8775(x - \underset{}{\overset{\pi/6=}{.5236}}) - .2218(x - .5236)^2 - .2276(x - .5236)^3}$$

$$s_1(.5236) = .5 \qquad S_1\left(\frac{\pi}{3}\right) = S_1(1.0472) = .8660$$

For $k=2$

$m_0 = 0$
$m_3 = 0$

$$S_{2,0} = y_2 = .8660$$

$$S_{2,1} = d_2 - \frac{h_2(2m_2 + m_3)}{6}$$

$$= .2559 - \frac{.5236(2(-1.1585)+0)}{6}$$

$$= .4551$$

$$S_{2,2} = \frac{m_2}{2} = \frac{-1.1585}{2} = -.5793$$

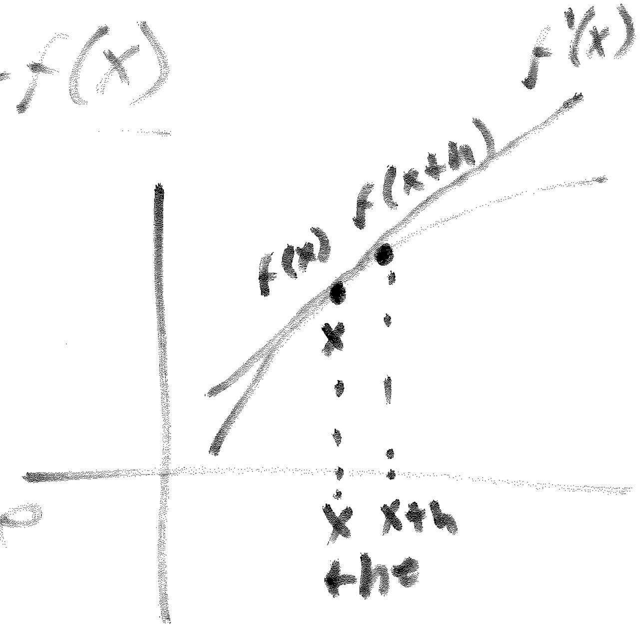
$$S_{2,3} = \frac{m_3 - m_2}{6h_2} = \frac{0-(-1.1585)}{6(.5236)} = .3688$$

$$S_2(x) = .8660 + .4551(x - \underset{\pi/3}{1.0472}) - .5793(x-1.0472)^2 + .3688(x-1.0472)^3$$

Numerical Diferentiation

We have from the definition of a derivative that:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



we can use the same definition to obtain an approximation to the derivative.

$$D_k = \frac{f(x+h_k) - f(x)}{h_k}$$

how small h_k should be?

h_k	D_k
.1	2.8588
.01	2.7319
.001	2.7196
.0001	2.7184
.00001	2.7183

$f(x) = e^x$

$f'(x) = e^x$

Let $x = 1$

$f'(1) = e^1 =$
$2.71828182\ldots$

$$f'(x) \approx D(x) = \frac{f(x+h) - f(x)}{h}$$

$$= \frac{e^{x+h} - e^x}{h}$$

$h = .1 \rightarrow f'(1) \approx \frac{e^{1+.1} - e^{1}}{.1} = 2.8588$

$h = .01 \rightarrow f'(1) \approx \frac{e^{1+.01} - e^1}{.01} = 2.7319$

We will see better approximations ~~ox~~ of the ~~than~~ derivative than this equation.

Central Differences Formula

Also called formula of order $O(h^2)$

This is a more accurate way of approximating the derivative. Using Taylor expansion of $f(x+h)$ and $f(x-h)$

$$① \quad f(x+h) = f(x) + f'(x)h + \frac{f''(x)h^2}{2!} + \underbrace{\frac{f'''(c_1)h^3}{3!}}_{\text{error}}$$

$$② \quad f(x-h) = f(x) - f'(x)h + \frac{f''(x)h^2}{2!} - \underbrace{\frac{f'''(c_2)h^3}{3}}_{\text{error}}$$

subtracting ① - ②

$$f(x+h) - f(x-h) = \cancel{f(x)} + f'(x)h + \cancel{\frac{f''(x)h^2}{2!}} + \frac{f'''(c_1)h^3}{3!}$$

$$- \cancel{f(x)} + f'(x)h - \cancel{\frac{f''(x)h^2}{2!}} + \frac{f'''(c_2)h^3}{3!}$$

$$f(x+h) - f(x-h) = 2f'(x)h + \underbrace{\frac{f'''(c_1) + f'''(c_2)}{3!}h^3}_{\text{error}}$$

$$f(x+h)-f(x-h) \approx 2f'(x)h$$

$$\boxed{f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}}$$

Example. $f(x)=e^x \quad h=.0001$

$$f'(1) = \frac{e^{1+.0001} - e^{1-.0001}}{2(.0001)}$$

$$= 2.7183$$

Let $h = .1$

$$f'(1) = \frac{e^{1+.1} - e^{1-.1}}{2(.1)}$$

$$= \frac{3.004166 - 2.459603}{.2}$$

$$= 2.722815$$

Compare to 2.8588 using $\frac{f(x+h)-f(x)}{h}$ or 2.718281 exact val

Centered formula of order $O(h^4)$ (189)

Do taylor expansion

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)h^2}{2!} + \frac{f'''(x)}{3!}h^3 + \frac{f^{IV}(x)h^4}{4!} + \frac{f^{V}(c)h^5}{5!}$$ error

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)h^2}{2!} - \frac{f'''(x)h^3}{3!} + \frac{f^{IV}(x)h^4}{4!} - \frac{f^{V}(c_2)h^5}{5!}$$ error

(1)

$$f(x+h) - f(x-h) = 2f'(x)h + \frac{2f'''(x)}{3!}h^3 + \frac{f^{V}(c_1)h^5}{5!} + \frac{f^{V}(c_2)h^5}{5!}$$

Using step size 2h instead of h in ①

$$② \; f(x+2h) - f(x-2h) = 4hf'(x) + 16h\frac{f'''(x)}{3!} + \overbrace{\frac{f^{V}(c_1)32h^5}{5!} + \frac{f^{V}(c_2)32h^5}{5!}}^{\text{error}}$$

Now multiply ① by 8 and subtract ② from it.

$$8f(x+h) - 8f(x-h) - f(x+2h) + f(x-2h) =$$

$$16f'(x)h + \cancel{16\frac{f'''(x)}{3!}h^3} + \frac{8f^{V}(c_1)h^5}{5!} + \frac{8f^{V}(c_2)h^5}{5!}$$

$$- 4hf'(x) - \cancel{16h\frac{f'''(x)}{3!}} - \frac{32f^{V}(c_1)h^5}{5!} - \frac{f^{V}(c_2)32h^5}{5!}$$

$$-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h) =$$

$$12f'(x)h + Kh^5 \leftarrow \text{error}$$

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$$f'(x) = \frac{-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)}{12h} - \underbrace{\frac{kh^5}{k}}_{err}$$

$$\boxed{f'(x) \cong \frac{-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)}{12h}}$$

$h = .001 \to 10^{-3}$ $(10^{-3})^4 = 10^{-12}$

$h^4 = 00\,00\,00\,00\,00\,00\,01$

$$\underline{\text{Error} = K_2 h^4}$$

$$\text{Error} = O(h^4)$$

Let $f(x) = \sin(x)$, $h = .001$, centered formula $O(h^4)$

$$f'\left(\frac{\pi}{3}\right) \approx \frac{-\sin\left(\frac{\pi}{3}+.002\right)+8\sin\left(\frac{\pi}{3}+.001\right) - 8\sin\left(\frac{\pi}{3}-.001\right)+\sin\left(\frac{\pi}{3}-.002\right)}{12(.001)}$$

$$\approx \frac{-.8670+6.9322-6.9242+.8650}{.012}$$

$$\approx .5$$

Exact val

$f(x) = \sin(x)$

$f'(x) = \cos(x)$

$f'\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = .5$

Using centered formula $O(h^2)$ 192

$$f'(x) \doteq \frac{\sin(\frac{\pi}{3}+.001)-\sin(\frac{\pi}{3}-.001)}{.002}$$

$h^2=(10^{-3})^2=10^{-6}$

$.000001$

$= .49999917$

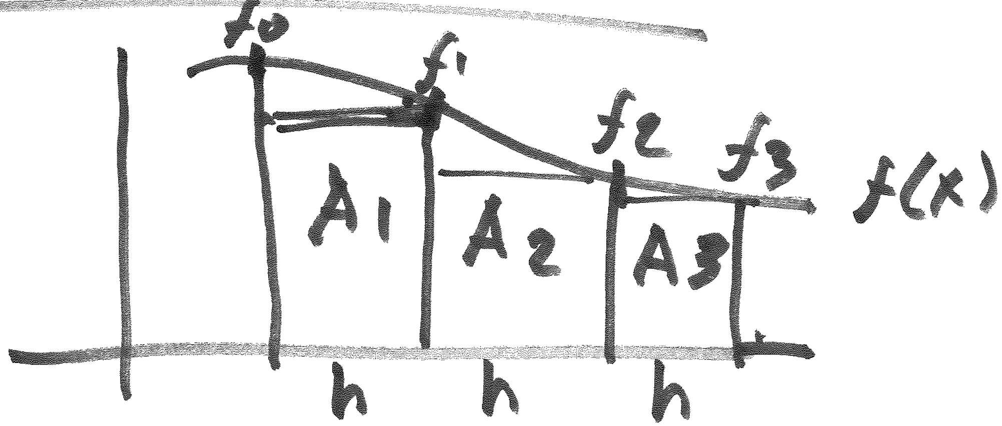
Using difference quotient

$$f'(x) = \frac{\sin(\frac{\pi}{3}+.001)-\sin(\frac{\pi}{3})}{.001}$$

$= .49956604$

Numerical Integration

Trapezoidal Rule

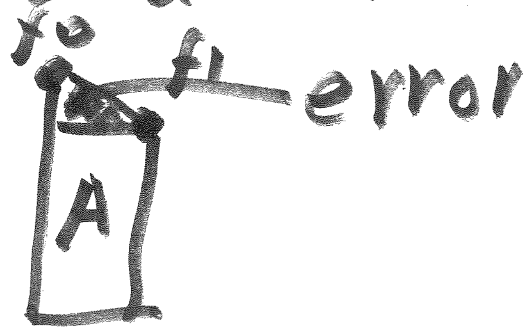


$$A = A_1 + A_2 + A_3$$

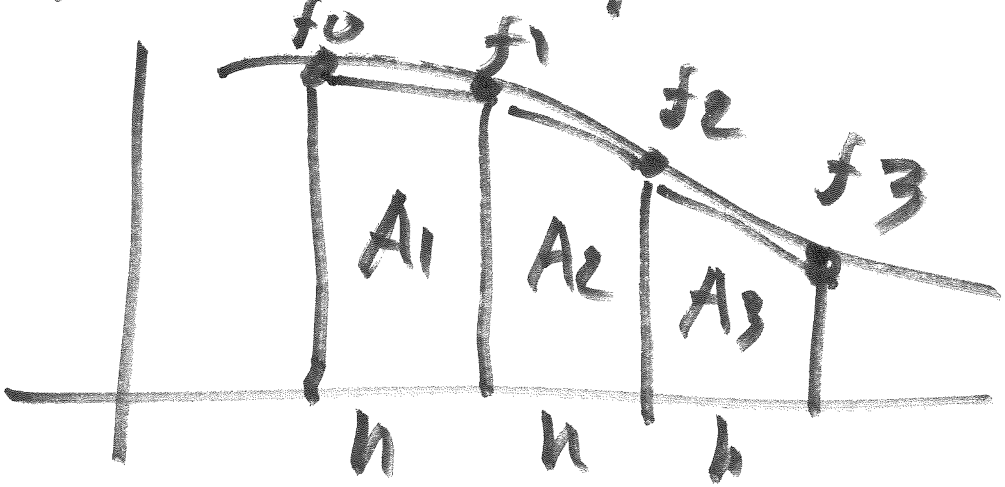
If we approximate the area using rectangles:

$$A \approx hf_1 + hf_2 + hf_3 = h(f_1 + f_2 + f_3)$$
$$= h\left(\sum_{i=1}^{3} f_i\right)$$

however there is an error on top (or below) each rectangle



To obtain a better approximation we use trapezoids



$$A_1 = h\left(\frac{f_0+f_1}{2}\right) \quad A_2 = \left(\frac{f_1+f_2}{2}\right)h \quad A_3 = \left(\frac{f_2+f_3}{2}\right)h$$

$$A_1+A_2+A_3 = \frac{f_0+f_1}{2}h + \frac{f_1+f_2}{2}h + \frac{f_2+f_3}{2}h$$

$$= \frac{f_0}{2}h + f_1 h + f_2 h + \frac{f_3}{2}h$$

In general

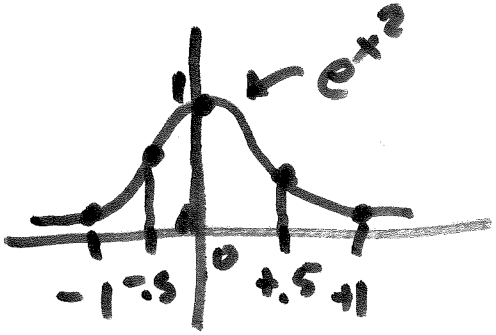
$$A = \frac{f_0}{2}h + f_1 h + f_2 h + f_3 h \cdots f_{m-1}h + \frac{f_m h}{2}$$

$$\boxed{\int_a^b f(x)\,dx = A = h\left[\frac{f(x_0)+f(x_m)}{2} + \sum_{i=1}^{m-1} f(x_i)\right]}$$

Trapezoidal Rule

$$\int_{-1}^{1} e^{-x^2}\,dx \approx .5\left[\frac{e^{-(-1)^2}+e^{-(1)^2}}{2} + e^{-(-.5)^2}+e^{-(0)^2}+e^{-(.5)^2}\right] \quad 95$$

$$\approx 1.4627$$



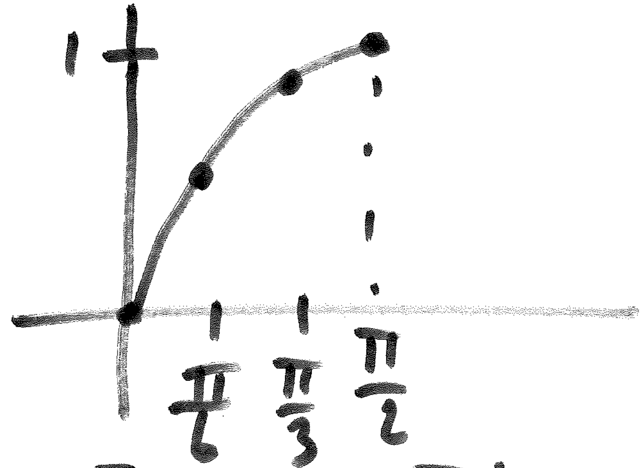
$h = .5$

$\approx$

$\approx$

Example

$$\int_0^{\frac{\pi}{2}} \sin(x)$$



$M = 3$

$h = \frac{\pi}{6}$

$$\int_0^{\frac{\pi}{2}} \sin(x) \approx \frac{\pi}{6}\left[\frac{0+\sin(\frac{\pi}{2})}{2} + \sin(\tfrac{\pi}{6}) + \sin(\tfrac{\pi}{3})\right]$$

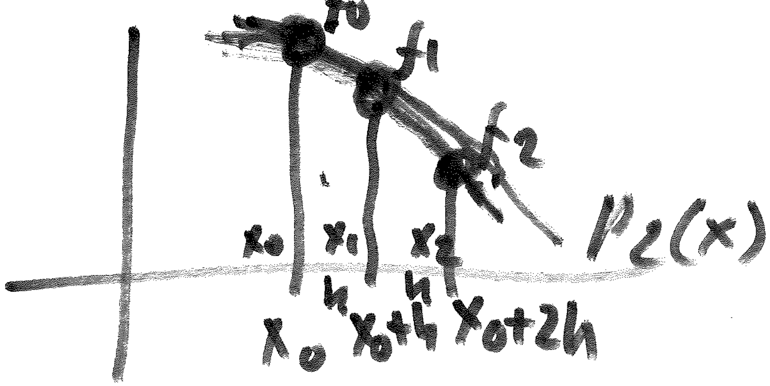
$$\approx .9770486$$

Exact Solution

$$\int_0^{\pi/2} \sin(x) = -\cos(x)\Big|_0^{\frac{\pi}{2}} = -\left[\cos(\tfrac{\pi}{2}) - \cos(0)\right] = 1$$

Simpson Rule

It approximates the integral of a function using a quadratic polynomial every 3 points



Using Lagrange Polynomials to obtain $P_2(x)$

$$P_2(x) = f_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$\int_{x_0}^{x_2} p_2(x)\,dx = \int_{x_0}^{x_2} f_0 \frac{(x-x_1)(x-x_2)}{(-h)(-2h)} + f_1 \frac{(x-x_0)(x-x_2)}{(h)(-h)}$$

$$+ f_2 \frac{(x-x_0)(x-x_1)}{(2h)(h)} \, dx$$

Lets do a change of variables Let $x = x_0 + ht$ $\qquad t = \frac{x-x_0}{h}$

$dx = h\,dt$

$$\int_{x_0}^{x_2} p_2(x)\,dx = \int_0^2 f_0 \frac{(\overset{-h+ht}{x_0+ht-x_1})(\overset{-2h+ht}{x_0+ht-x_2})}{(-h)(-2h)} h\,dt \qquad t_0 = \frac{x_0-x_0}{h} = 0 \qquad t_2 = \frac{x_2-x_0}{h} = \frac{2h}{h} = 2$$

$$+ \int_0^2 f_1 \frac{(\cancel{x_0}+ht-\cancel{x_0})(\overset{-2h+ht}{x_0+ht-x_2})}{(h)(-h)} h\,dt$$

$$+ \int_0^2 f_2 \frac{(\cancel{x_0}+ht-\cancel{x_0})(\overset{-h+ht}{x_0+ht-x_1})}{(2h)\,h} h\,dt$$

$$\int_{x_0}^{x_2} P_2(x)\,dx = \int_0^2 f_0 \frac{(-\cancel{h}+\cancel{h}t)(-2\cancel{h}+\cancel{h}t)}{2\cancel{h^2}} h\,dt$$

$$+ \int_0^2 f_1 \frac{\cancel{h}t(\cancel{h}t-2\cancel{h})}{-\cancel{h^2}} h\,dt$$

$$+ \int_0^2 f_2 \frac{\cancel{h}t(\cancel{h}t-\cancel{h})}{2\cancel{h^2}} h\,dt$$

$$\int_{x_0}^{x_2} P_2(x)\,dx = \int_0^2 f_0 \frac{(t-1)(t-2)}{2} h\,dt$$

$$+ \int_0^2 f_1 \frac{(t)(t-2)}{(-1)} h\,dt$$

$$+ \int_0^2 f_2 \frac{t(t-1)}{2} h\,dt$$

$$\int_{x_0}^{x_2} p_2(x)\,dx = \frac{f_0}{2} h \int_0^2 t^2 - 3t + 2 \, dt$$

$$+ f_1 h \int_0^2 t^2 - 2t \, dt$$

$$+ \frac{f_2 h}{2} \int_0^2 t^2 - t \, dt$$

$$\int_{x_0}^{x_2} p_2(x)\,dx = \frac{f_0}{2} h \left[\frac{t^3}{3} - \frac{3t^2}{2} + 2t \right]_0^2$$

$$- f_1 h \left[\frac{t^3}{3} - \frac{2t^2}{2} \right]_0^2$$

$$+ \frac{f_2 h}{2} \left[\frac{t^3}{3} - \frac{t^2}{2} \right]_0^2$$

$$\int_{x_0}^{x_2} P_2(x)\,dx = \frac{f_0}{2}h\left[\frac{8}{3}-\frac{12}{2}+4-0-0-0\right] \quad \frac{8}{3}-6+4=\frac{8}{3}-2=\frac{8-6}{3}=\frac{2}{3}$$

$$-f_1 h\left[\frac{8}{3}-\frac{8}{2}-0-0\right] \quad \frac{8}{3}-\frac{4}{1}=\frac{8-12}{3}=-\frac{4}{3}$$

$\frac{8}{3}-6$

$$+\frac{f_2 h}{2}\left[\frac{8}{3}-\frac{4}{2}-0-0\right]$$

$$\int_{x_0}^{x_2} P_2(x)\,dx = \frac{f_0}{\cancel{2}}h\frac{\cancel{2}}{3} - f_1 h\left(-\frac{4}{3}\right) + \frac{f_2 h}{\cancel{2}}\left(\frac{2}{3}\right)$$

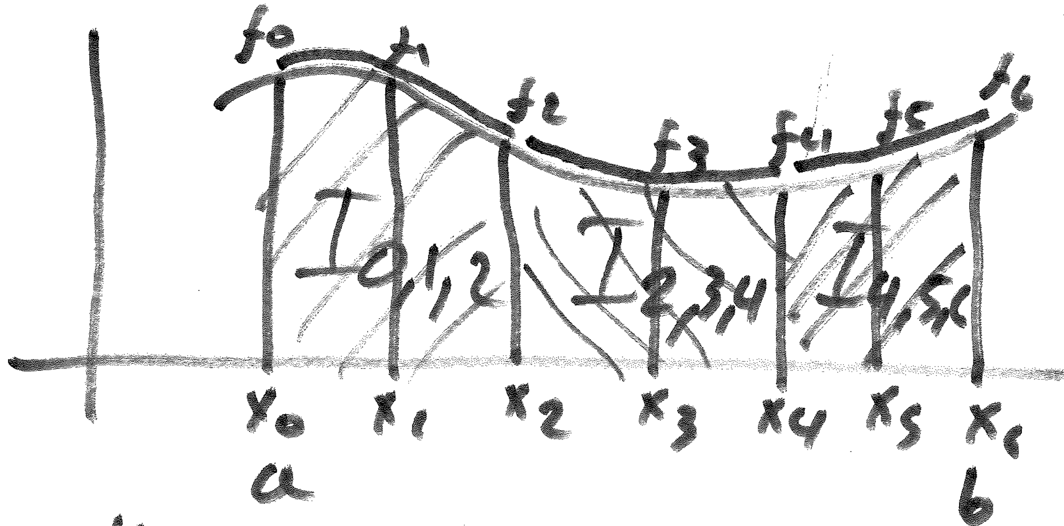
$$\frac{8}{3}-\frac{2}{1} = \frac{8-6}{3}$$

$$\int_{x_0}^{x_2} P_2(x)\,dx = \frac{f_0 h}{3} + f_1 h\frac{4}{3} + \frac{f_2 h}{3}$$

$$\boxed{\int_{x_0}^{x_2} P_2(x)\,dx = \frac{h}{3}\left[f_0+4f_1+f_2\right]}$$

Simpson Rule

Now if the interval $[a, b]$ is subdivided into $2M$ subintervals $[x_k, x_{k+1}]$ of equal width $h = \frac{b-a}{2M}$



$2M = 6$

$x_0, x_1 \cdots x_{2M}$

we will have M quadratic segments $M = \frac{6}{2} = 3$

$$I = I_{0,1,2} + I_{2,3,4} + I_{4,5,6}$$

$$= \frac{h}{3}\left[f_0 + 4f_1 + f_2\right] + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right] + \frac{h}{3}\left[f_4 + 4f_5 + f_6\right]$$

$$I = \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + 4f_5 + f_6\right]$$

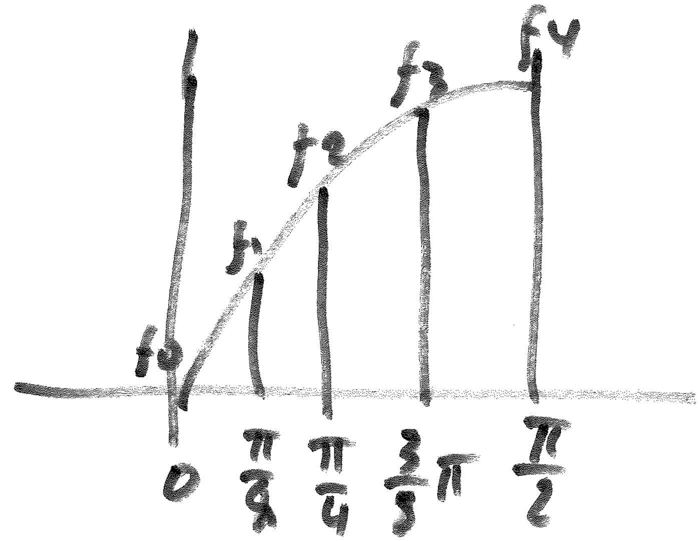
In General

$$\boxed{I = \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots 4f_{2M-1} + f_{2M}\right]}$$

Simpson Rule

Example

$$\int_0^{\pi/2} \sin(x)$$



$2M = 4$ a b

$$h = \frac{\frac{\pi}{2} - 0}{2M \to 4} = \frac{\pi}{8}$$

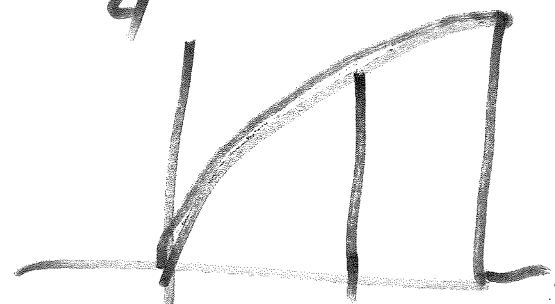
$$I = \frac{\pi/8}{3}\left[\sin(0) + 4\sin\left(\frac{\pi}{8}\right) + 2\sin\left(\frac{\pi}{4}\right) + 4\sin\left(\frac{3}{8}\pi\right) + \sin\left(\frac{\pi}{2}\right)\right]$$

$$= 1.00013458$$

Exact value $= 1$

With $2M = 2$

$$h = \frac{\frac{\pi}{2} - 0}{2} = \frac{\pi}{4}$$



$$I = \frac{\pi/4}{3}\left[\sin(0) + 4\sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{2}\right)\right]$$

$$= 1.002279878$$