

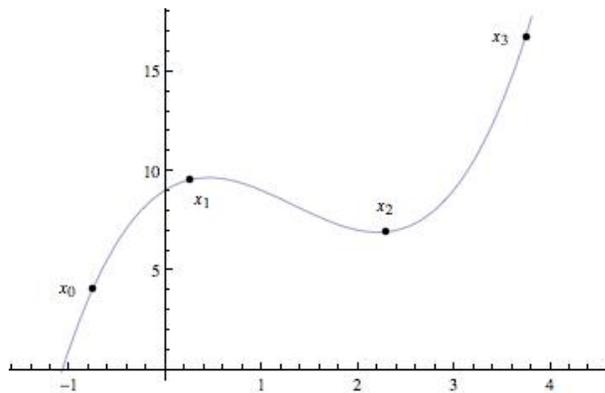
CS NOTES: 11-15 JULY

NOTES BY: ANIKA ROUNDS

1. 11 JULY 2011

- Last time we talked about splines and their properties. We found that there are 3 properties that must be met:
 - The equation must pass through every given point
 - The function must be continuous (where one section ends, the other must begin)
 - The function must be continuous in the first and second derivatives
- For each polynomial $s_k(x)$ there are four coefficients to be determined $s_{k,0}, s_{k,1}, s_{k,2}, s_{k,3}$ (four degrees of freedom)

$$s_k(x) = s_{k,0} + s_{k,1}(x - x_k) + s_{k,2}(x - x_k)^2 + s_{k,3}(x - x_k)^3$$



- There are n polynomials ($n + 1$ data points)
- Each polynomial has 4 coefficients to be determined
 - $4n$ coefficients in total
- From II (each polynomial passes through the $n + 1$ data points)
- From III, IV, V each supplies $n - 1$ constraints, so in total II, III, IV, V give
$$n + 1 + 3(n - 1) = 4n - 2 \text{ constraints}$$
- So since we have to compute $4n$ coefficients, that leaves 2 additional degrees of freedom

- Since $s_k(x)$ is cubic, the second derivative is a line we can use lagrange to obtain the equation of the second derivative

$$\begin{aligned}
& - s_k''(x) = s''(x_k) \frac{(x-x_{k+1})}{(x_k-x_{k+1})} + s''(x_{k+1}) \frac{(x-x_k)}{(x_{k+1}-x_k)} \\
& - \text{So } s_k''(x) = s''(x_k) \text{ when } x = x_k \\
& - \text{So } s_k''(x) = s''(x_{k+1}) \text{ when } x = x_{k+1} \\
& - \text{Use } m_k = s''(x_k), m_{k+1} = s''(x_{k+1}) \text{ and } h_k = x_{k+1} - x_k \\
& - s_k''(x) = m_k \frac{(x_{k+1}-x)}{h_k} + m_{k+1} \frac{(x-x_k)}{h_k} \\
& - \text{integrate } s_k''(x) \text{ twice}
\end{aligned}$$

$$\begin{aligned}
s_k(x) &= \int_{x_k}^{x_{k+1}} \int_{x_k}^{x_{k+1}} s_k''(x) dx dx \\
&= \int_{x_k}^{x_{k+1}} \int_{x_k}^{x_{k+1}} \left(m_k \frac{(x_{k+1}-x)}{h_k} + m_{k+1} \frac{(x-x_k)}{h_k} \right) dx dx \\
&= \int_{x_k}^{x_{k+1}} \left(\frac{m_k}{2h_k} (x_{k+1}-x)^2 + \frac{m_{k+1}}{2h_k} (x-x_k)^2 + c_1 \right) dx \\
&= \frac{m_k}{6h_k} (x_{k+1}-x)^3 + \frac{m_{k+1}}{6h_k} (x-x_k)^3 + c_1 x + c_2
\end{aligned}$$

- We can express $c_1 x + c_2$ as follows (by change of constants)

$$c_1 x + c_2 = p_k * (x_{k+1} - x) + q_k * (x - x_k)$$

- So we have

$$s_k(\mathbf{x}) = \frac{\mathbf{m}_k}{6\mathbf{h}_k} (\mathbf{x}_{k+1} - \mathbf{x})^3 + \frac{\mathbf{m}_{k+1}}{6\mathbf{h}_k} (\mathbf{x} - \mathbf{x}_k)^3 + \mathbf{p}_k * (\mathbf{x}_{k+1} - \mathbf{x}) + \mathbf{q}_k * (\mathbf{x} - \mathbf{x}_k) \quad (1)$$

- Substituting x_k , x_{k+1} , and $s_k(x) = y_k$, $s_{k+1} = y_{k+1}$ in (1)

$$\begin{aligned}
s_k(x_k) &= y_k = \frac{m_k}{6h_k} (x_{k+1} - x_k)^3 + \frac{m_{k+1}}{6h_k} (x_k - x_k)^3 \\
&\quad + p_k * (x_{k+1} - x_k) + q_k * (x_k - x_k) \\
y_k &= \frac{m_k}{6h_k} (x_{k+1} - x_k)^3 + p_k * (x_{k+1} - x_k) \\
y_k &= \frac{m_k}{6h_k} h_k^3 + p_k h_k \\
\mathbf{y}_k &= \frac{\mathbf{m}_k}{6} \mathbf{h}_k^2 + \mathbf{p}_k \mathbf{h}_k \quad (\text{a}) \\
s_k(x_{k+1}) &= y_{k+1} = \frac{m_k}{6h_k} (x_{k+1} - x_{k+1})^3 + \frac{m_{k+1}}{6h_k} (x_{k+1} - x_k)^3 \\
&\quad + p_k * (x_{k+1} - x_{k+1}) + q_k * (x_{k+1} - x_k) \\
y_{k+1} &= \frac{m_{k+1}}{6h_k} (x_{k+1} - x_k)^3 + q_k * (x_{k+1} - x_k) \\
y_{k+1} &= \frac{m_{k+1}}{6h_k} h_k^3 + q_k h_k \\
\mathbf{y}_{k+1} &= \frac{\mathbf{m}_{k+1}}{6} \mathbf{h}_k^2 + \mathbf{q}_k \mathbf{h}_k \quad (\text{b})
\end{aligned}$$

- From (a), obtain

$$\mathbf{p}_k = \frac{\mathbf{y}_k}{\mathbf{h}_k} - \frac{\mathbf{m}_k}{6} \mathbf{h}_k \quad (2)$$

- From (b), obtain

$$\mathbf{q}_k = \frac{\mathbf{y}_{k+1}}{\mathbf{h}_k} - \frac{\mathbf{m}_{k+1}}{6} \mathbf{h}_k \quad (3)$$

– Substituting (2) and (3) into (1),

$$s_k(x) = \frac{m_k}{6h_k}(x_{k+1}-x)^3 + \frac{m_{k+1}}{6h_k}(x-x_k)^3 + \left(\frac{y_k}{h_k} - \frac{m_k}{6}h_k\right)(x_{k+1}-x) + \left(\frac{y_{k+1}}{h_k} - \frac{m_{k+1}}{6}h_k\right)(x-x_k)$$

– Only the terms m_k and m_{k+1} are unknown.

– To obtain these values we differentiate $s_k(x)$

$$s'_k(\mathbf{x}) = -\frac{\mathbf{m}_k}{2\mathbf{h}_k}(\mathbf{x}_{k+1}-\mathbf{x})^2 + \frac{\mathbf{m}_{k+1}}{2\mathbf{h}_k}(\mathbf{x}-\mathbf{x}_k)^2 - \left(\frac{y_k}{h_k} - \frac{m_k}{6}h_k\right) + \left(\frac{y_{k+1}}{h_k} - \frac{m_{k+1}}{6}h_k\right) \quad (4)$$

$$s'_k(x_k) = -\frac{m_k}{2h_k}(x_{k+1}-x_k)^2 + \frac{m_{k+1}}{2h_k}(x_k-x_k)^2 - \left(\frac{y_k}{h_k} - \frac{m_k}{6}h_k\right) + \left(\frac{y_{k+1}}{h_k} - \frac{m_{k+1}}{6}h_k\right)$$

$$s'_k(x_k) = -\frac{m_k}{2h_k}(x_{k+1}-x_k)^2 - \left(\frac{y_k}{h_k} - \frac{m_k}{6}h_k\right) + \left(\frac{y_{k+1}}{h_k} - \frac{m_{k+1}}{6}h_k\right)$$

$$s'_k(x_k) = -\frac{m_k h_k}{2} - \frac{y_k}{h_k} + \frac{m_k h_k}{6} + \frac{y_{k+1}}{h_k} - \frac{m_{k+1} h_k}{6}$$

$$s'_k(x_k) = -\frac{m_k h_k}{3} - \frac{y_k}{h_k} + \frac{y_{k+1}}{h_k} - \frac{m_{k+1} h_k}{6}$$

– Let $\mathbf{d}_k = \frac{y_{k+1}-y_k}{h_k}$ (c)

$$s'_k(\mathbf{x}_k) = -\frac{\mathbf{m}_k \mathbf{h}_k}{3} - \frac{\mathbf{m}_{k+1} \mathbf{h}_k}{6} + \mathbf{d}_k \quad (5)$$

– Substituting k by $k-1$ in (4)

$$s'_{k-1}(x) = -\frac{m_{k-1}}{2h_{k-1}}(x_k-x)^2 + \frac{m_k}{2h_{k-1}}(x-x_{k-1})^2 - \left(\frac{y_{k-1}}{h_{k-1}} - \frac{m_{k-1}}{6}h_{k-1}\right) + \left(\frac{y_k}{h_{k-1}} - \frac{m_k}{6}h_{k-1}\right)$$

$$s'_{k-1}(x_k) = -\frac{m_{k-1}}{2h_{k-1}}(x_k-x_k)^2 + \frac{m_k}{2h_{k-1}}(x_k-x_{k-1})^2 - \left(\frac{y_{k-1}}{h_{k-1}} - \frac{m_{k-1}}{6}h_{k-1}\right) + \left(\frac{y_k}{h_{k-1}} - \frac{m_k}{6}h_{k-1}\right)$$

$$s'_{k-1}(x_k) = \frac{m_k h_{k-1}}{3} + \frac{m_{k-1}}{6}h_{k-1} + \frac{y_k - y_{k-1}}{h_{k-1}}$$

– Let $\mathbf{d}_{k-1} = \frac{y_k - y_{k-1}}{h_{k-1}}$ (d)

$$s'_{k-1}(\mathbf{x}_k) = \frac{\mathbf{m}_k \mathbf{h}_{k-1}}{3} + \frac{\mathbf{m}_{k-1} \mathbf{h}_{k-1}}{6} + \mathbf{d}_{k-1} \quad (6)$$

2. 12 JULY 2011

Continuing from 11 July,

- Set (5) and (6) equal to one another,

$$-\frac{m_k h_k}{3} - \frac{m_{k+1} h_k}{6} + d_k = \frac{m_k h_{k-1}}{3} + \frac{m_{k-1} h_{k-1}}{6} + d_{k-1}$$

$$-2m_k h_k - m_{k+1} h_k + 6d_k = 2m_k h_{k-1} + m_{k-1} h_{k-1} + 6d_{k-1}$$

$$6(d_k - d_{k-1}) = +m_{k-1} h_{k-1} + 2m_k (h_k + h_{k-1}) + m_{k+1} h_k$$

- Let $u_k = 6(d_k - d_{k-1})$

$$\mathbf{u}_k = +\mathbf{m}_{k-1} \mathbf{h}_{k-1} + 2\mathbf{m}_k (\mathbf{h}_k + \mathbf{h}_{k-1}) + \mathbf{m}_{k+1} \mathbf{h}_k \quad (7)$$

- (7) gives $n - 1$ equations that can be used to solve m_0, m_1, \dots, m_n , but since we have $n + 1$ unknowns we need two more equations.
- These next two equations are used to evaluate m_0 and m_n :
 - Assume that $m_0 = s''(x_0)$ and $m_n = s''(x_n)$ are given.
 - Then from (7) at $k = 1$ we get

$$\mathbf{u}_1 = +\mathbf{m}_0 \mathbf{h}_0 + 2\mathbf{m}_1 (\mathbf{h}_1 + \mathbf{h}_2) + \mathbf{m}_2 \mathbf{h}_1 \quad (8)$$

– And at $k = n - 1$

$$\mathbf{u}_{n-1} = +\mathbf{m}_{n-2} \mathbf{h}_{n-2} + 2\mathbf{m}_{n-1} (\mathbf{h}_{n-1} + \mathbf{h}_{n-2}) + \mathbf{m}_n \mathbf{h}_{n-1} \quad (9)$$

- Since $m_0 = s''(x_0)$ and $m_n = s''(x_n)$ are given, with (7), (8), (9) we can determine m_1, \dots, m_{n-1}
- This gives a strictly diagonal dominant system

$$\begin{bmatrix} b_1 & c_1 & 0 & \cdots & \cdots & \cdots & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & \cdots & \vdots \\ 0 & a_3 & b_3 & c_3 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & a_{n-3} & b_{n-3} & c_{n-3} \\ 0 & \cdots & \cdots & \cdots & 0 & a_{n-2} & b_{n-2} \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ \vdots \\ \vdots \\ m_{n-1} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ \vdots \\ u_{n-1} \end{bmatrix}$$

- Other end point constraints to choose m_0 and m_n :

1) Clamped spline

– The first derivatives at the end points ($s'(x_0)$ and $s'(x_n)$) are given.

$$h_{k-1} m_{k-1} + 2(h_{k-1} + h_k) m_k + h_k m_{k+1} = u_k$$

2) Natural spline

- The second derivatives at the ends are given and are zero.

$$2(h_0 + h_1)m_1 + h_1m_2 = u_1$$

3) Extrapolated spline

- s_0 and s_1 are the same spline, s_{n-2} and s_{n-1} are also the same spline.

- See equation in the book.

4) Parabolic terminated spline

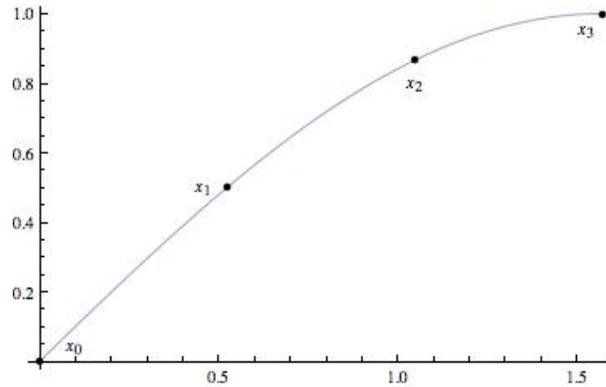
- s_0 and s_{n-1} are quadratic instead of cubic.

5) End point curvature adjusted spline

- $s''(x_0)$ and $s''(x_n)$ are given.

- Example. Find the natural spline that approximates $y = \sin(x)$ in the interval $(0, \frac{\pi}{2})$ with the four points equally spaced.

k	x_0	y_0
0	0	0
1	$\frac{\pi}{6} = 0.5236$.5
2	$\frac{\pi}{3} = 1.0472$.8660
3	$\frac{\pi}{2} = 1.5708$	1



$$s''(x_0) = 0 = s''(x_3)$$

$$m_0 = 0 = m_3$$

$$h_k = x_{k+1} - x_k \implies h_0 = h_1 = h_2 = \frac{\pi}{6}$$

$$d_k = \frac{y_{k+1} - y_k}{h_k}$$

$$d_0 = \frac{y_1 - y_0}{h_0} = \frac{.5 - 0}{\frac{\pi}{6}} = 0.95493$$

$$d_1 = \frac{y_2 - y_1}{h_1} = \frac{.866 - .5}{\frac{\pi}{6}} = 0.699009$$

$$d_2 = \frac{y_3 - y_2}{h_2} = \frac{1 - .866}{\frac{\pi}{6}} = 0.255921$$

$$u_k = 6(d_k - d_{k-1})$$

$$u_1 = 6(d_1 - d_0) = 6(0.699009 - 0.95493) = -1.53553$$

$$u_2 = 6(d_2 - d_1) = 6(0.255921 - 0.699009) = -2.65853$$

From natural spline,

$$2(h_0 + h_1)m_1 + h_1m_2 = u_1 \quad (10)$$

Substituting h_0, h_1, h_1, u_1, u_2 in (10),

$$2\left(\frac{\pi}{6} + \frac{\pi}{6}\right)m_1 + \frac{\pi}{6}m_2 = u_1$$

$$2.0944m_1 + .5236m_2 = -1.5354$$

$$\frac{\pi}{6}m_1 + 2\left(\frac{\pi}{6} + \frac{h}{6}\right)m_2 = u_2$$

$$.5236m_1 + 2.0944m_2 = -2.0944$$

$$\begin{bmatrix} 2.0944 & .5236 \\ .5236 & 2.0944 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} -1.5354 \\ -2.6586 \end{bmatrix}$$

$$\begin{bmatrix} 2.0944 & .5236 \\ 0 & 1.9635 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} -1.5354 \\ -2.2749 \end{bmatrix}$$

$$m_2 = -1.1585$$

$$m_1 = -.4435$$

3. 13 JULY 2011

How do we build the splines from m_1 and m_2 ? We have the spline:

$$s_k(x) = s_{k,0} + s_{k,1}(x - x_k) + s_{k,2}(x - x_k)^2 + s_{k,3}(x - x_k)^3 \quad (f)$$

We need to compute $s_{k,0}, s_{k,1}, s_{k,2}, s_{k,3}$ from m_1, m_2, d_k, u_k , etc.

$$y_0 = s_0(x_0) = s_{0,0} + s_{0,1}(x_0 - x_0) + s_{0,2}(x_0 - x_0)^2 + s_{0,3}(x_0 - x_0)^3$$

$$y_0 = s_{0,0}$$

$$s'_k(x) = s_{k,1} + 2s_{k,2}(x - x_k) + 3s_{k,3}(x - x_k)^2 \quad (g)$$

$$s''_k(x) = 2s_{k,2} + 6s_{k,3}(x - x_k) \quad (h)$$

From the definition of m_k ,

$$m_k = s''(x_k) \quad (i)$$

Substituting x_k in (h),

$$s_{k,2} = \frac{s_k''(x_k)}{2} = \frac{m_k}{2} \quad (j)$$

From (g),

$$s_k'(x) = s_{k,1} + 2s_{k,2}(x - x_k) + 3s_{k,3}(x - x_k)^2$$

Substitute $s_k'(x_k) = s_{k,1}$ and (5),

$$s_{k,1} = -\frac{m_k h_k}{3} - \frac{m_{k+1} h_k}{6} + d_k$$

Making $x = x_{k+1}$ in $s_k''(x)$ in (h),

$$s_k''(x_{k+1}) = 2s_{k,2} + 6s_{k,3}(x_{k+1} - x_k)$$

$$m_{k+1} = 2\left(\frac{m_k}{2}\right) + 6s_{k,3}h_k$$

$$s_{k,3} = \frac{m_{k+1} - m_k}{6h_k} \quad (o)$$

Thus,

$$s_{k,0} = y_k$$

$$s_{k,1} = -\frac{m_k h_k}{3} - \frac{m_{k+1} h_k}{6} + d_k$$

$$s_{k,2} = \frac{s_k''(x_k)}{2} = \frac{m_k}{2}$$

$$s_{k,3} = \frac{m_{k+1} - m_k}{6h_k}$$

$$\begin{bmatrix} s_{0,0} & s_{0,1} & s_{0,2} & s_{0,3} \\ s_{1,0} & s_{1,1} & s_{1,2} & s_{1,3} \\ s_{2,0} & s_{2,1} & s_{2,2} & s_{2,3} \end{bmatrix} = \begin{bmatrix} 0 & .9936 & 0 & -.1412 \\ .5 & .8775 & -.2218 & -.2276 \\ .8660 & .4581 & -.5793 & .3688 \end{bmatrix}$$

$$s_0(x) = .9936(x - x_0) - .1412(x - x_0)^3$$

$$s_1(x) = .5 + .8775(x - x_1) - .2218(x - x_1)^2 - .2276(x - x_1)^3$$

$$s_2(x) = .8660 + .4581(x - x_2) - .5793(x - x_2)^2 + .3688(x - x_2)^3$$

One way you can check if your spline is correct is by trying,

$$s_k(x_{k+1}) = s_{k+1}(x_{k+1})$$

$$s_k'(x_{k+1}) = s_{k+1}'(x_{k+1})$$

$$s_k''(x_{k+1}) = s_{k+1}''(x_{k+1})$$

4. 14-15 JULY 2011

Exam.