CS-314 Summer-2011

PADE APPROXIMATIONS

- Rational approximations for functions over a small portion of its domain.
- Rational approximations have a smaller overall error than polynomial approximations over a given interval & computational effort.

Conditions:

- The function to be approximated, $f(x)$ and its first derivative $f'(x)$ be continuous at $x = 0$.

  Why $x=0$?

  - Simpler computation
  - Function can be approximated at any value by suitable change of variables.

    Eg: $f(x)$ to be approximated at $x = x_1 \neq 0$.
    We make the substitution, $t = x - x_1$ into $f(x)$ such that $f(x) = f(t + x_1)$. Now, $f(t+x_1)$ is approximated at $t=0$, & then by substituting for t, we get the solution for $f(x)$.

- We wish to approximate $f(x)$ in the form:

$$f(x) \approx R_{N,M}(x) = \frac{P_N(x)}{Q_M(x)} \quad \text{for } a \leq x \leq b.$$

where $P_N(x) = P_0 + P_1 x + P_2 x^2 + \dots + P_N x^N$ (Degree-N)

& $Q_M(x) = \underline{1} + q_1 x + q_2 x^2 + \dots + q_M x^M$ (Degree-M)

Note: the constant term in $Q_M(x)$ is used without loss of generality to show it is always non-zero.

In the case, $Q_M(x) = 1$, then $P_N(x)$ is the same as the Maclaurin' Series for $f(x)$.

The best Pade approximation is attained when:

$\Rightarrow M = N$ ie. when P & Q polynomials have the same degree

(or)

$\Rightarrow N = M+1$ ie. P has one degree more than Q.

Method:

If $f(x)$ is analytic over an interval $[a, b]$ & has the Maclaurin series of the following form:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_k x^k + \dots$$

we wish to approximate,

$$f(x) \approx R_{N,M}(x) = \frac{P_N(x)}{Q_M(x)}.$$

ie. $f(x) \cdot Q_M(x) \approx P_N(x)$

ie. $f(x) \cdot Q_M(x) - P_N(x) \approx 0.$

$$\Rightarrow \left(a_0 + a_1 x + \dots + a_k x^k + \dots\right)\left(1 + q_1 x + q_2 x^2 + \dots + q_M x^M\right) - \left(P_0 + P_1 x + \dots + P_N x^N\right) \approx 0$$

Expanding the terms & by comparing the coefficients to zero,

$$\left.\begin{aligned} a_0 - P_0 &= 0 \\ q_1 a_0 + a_1 - P_1 &= 0 \\ q_2 a_0 + q_1 a_1 + a_2 - P_2 &= 0 \\ &\vdots \\ q_M a_{N-M} + q_{M-1} a_{N-M+1} + \dots + a_N - P_N &= 0. \end{aligned}\right\} \begin{array}{l} N+1 \\ \text{equations} \end{array}$$

and

$$\left.\begin{aligned} q_M a_{N-M+1} + q_{M-1} a_{N-M+2} + \dots + q_1 a_N + a_{N+1} &= 0 \\ q_M a_{N-M+2} + q_{M-1} a_{N-M+3} + \dots + q_1 a_{N+1} + a_{N+2} &= 0 \\ &\vdots \\ q_M a_N + q_{M-1} a_{N+1} + \dots + q_1 a_{N+M-1} + a_{N+M} &= 0 \end{aligned}\right\} \text{M equations.}$$

We solve the 2^{nd} set of M equations for $q_1, q_2, \dots, q_M$ and substitute these values into the 1^{st} N+1 equations to obtain $P_1, P_2, \dots, P_N$.

Example

Find the Pade approximation $R_{2,2}(x)$ for $f(x) = \frac{\ln(x+1)}{x}$ starting with the Maclaurin's expansion:

$$f(x) = \frac{\ln(1+x)}{x} \approx 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \dots$$

Solution:

Check for continuity of $f(0)$ & $f'(0)$.

$$f(x) = \frac{\ln(1+x)}{x}$$

$$f(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\frac{d}{dx}[\ln(1+x)]}{\frac{d}{dx}(x)} \quad \text{(By L'Hopital Rule)}$$

$$= \lim_{x \to 0} \frac{\frac{1}{1+x}}{1} = \lim_{x \to 0} \frac{1}{1+x} = 1 \Rightarrow \text{continuous at } x = 0.$$

$$f'(0) = \lim_{x \to 0} f'(x) = \lim_{x \to 0} \left[\frac{x - (1+x)\ln(1+x)}{x^2(1+x)}\right] = \lim_{x \to 0} \frac{1 - \ln(1+x) + \frac{1+x}{1+x}}{2x(1+x) + x^2}$$

$$= \lim_{x \to 0} \frac{1 - \ln(1+x) + 1}{2x + 3x^2} = \lim_{x \to 0} \frac{-1/1+x}{2 + 6x} = \frac{-1}{2}$$

$\Rightarrow f'(x)$ is continuous at $x = 0$.

$N=2,\ M=2$

$\Rightarrow P_2(x) = P_0 + P_1 x + P_2 x^2$

$\quad Q_2(x) = 1 + q_1 x + q_2 x^2$

$$\Rightarrow \left(1 - \frac{x}{2} + \frac{x^2}{3} \cdots\cdots\right) \approx \frac{P_0 + P_1 x + P_2 x^2}{1 + q_1 x + q_2 x^2}$$

$$\Rightarrow \left(1 - \frac{x}{2} + \frac{x^2}{3} \cdots\cdots\right)\left(1 + q_1 x + q_2 x^2\right) - \left(P_0 + P_1 x + P_2 x^2\right) \approx 0.$$

$\cancel{\Rightarrow 1 + q_1 x +}$

$$\Rightarrow 1 + \left(q_1 - \frac{1}{2}\right)x + \left(q_2 - \frac{q_1}{2} + \frac{1}{3}\right)x^2 - \left(\frac{q_2}{2} - \frac{q_1}{3} + \frac{1}{4}\right)x^3$$
$$+ \left(\frac{q_2}{3} - \frac{q_1}{4} + \frac{1}{5}\right)x^4 - \left(P_0 + P_1 x + P_2 x^2\right) \approx 0.$$

By comparing co-efficients,

	Co-efficient of
$1 - P_0 = 0$	x^0
$-\frac{1}{2} + q_1 - P_1 = 0$	x^1
$-\frac{q_1}{2} + \frac{1}{3} + q_2 - P_2 = 0$	x^2
$-\frac{1}{4} + \frac{q_1}{3} - \frac{q_2}{2} = 0$	x^3
$-\frac{q_1}{4} + \frac{1}{5} + \frac{q_2}{3} = 0$	x^4

$\Rightarrow q_2 = 3\left(\frac{q_1}{4} - \frac{1}{5}\right)$ from last equation.

Sub. into 2nd last equation,

$$-\frac{1}{4} + \frac{q_1}{3} - \frac{3q_1}{8} + \frac{3}{10} = 0$$

$$\cancel{\Rightarrow \frac{-8q_1}{24} = \frac{-1}{8} \Rightarrow q_1 = \frac{3}{5}}$$

$$\Rightarrow \frac{q_1}{24} = \frac{1}{20} \Rightarrow q_1 = \frac{6}{5}$$

$$\Rightarrow q_2 = 3\left(\frac{6}{20} - \frac{1}{5}\right) = \frac{3}{10}$$

Sub. into 2nd, 3rd equations,

$$P_2 = q_2 - \frac{q_1}{2} + \frac{1}{3} = \frac{3}{10} - \frac{6}{10} + \frac{1}{3} = \frac{1}{3} - \frac{3}{10} = \frac{1}{30}$$

$$P_1 = q_1 - \frac{1}{2} = \frac{6}{5} - \frac{1}{2} = 7/10$$

$$P_0 = 1$$

$$\Rightarrow f(x) = \frac{\ln(1+x)}{x} \approx \frac{1 + \frac{7x}{10} + \frac{x^2}{30}}{1 + \frac{6x}{5} + \frac{3x^2}{10}} = \frac{30 + 21x + x^2}{30 + 36x + 9x^2}$$

Chk: For $x=1$, $f(x) = \frac{\ln 2}{1} = 0.6931$

$$R_{2,2}(x) = \frac{30+21}{30+36+9} = \frac{51}{75} = 0.68 \approx 0.6931$$

$\boxed{\text{Verified}}$.

Given $R_{2,2}(x)$ for $f(x)$, we can find $\ln(1+x)$

$$\ln(1+x) = x \cdot R_{2,2}(x) = \frac{30x + 21x^2 + x^3}{30 + 36x + 9x^2}.$$