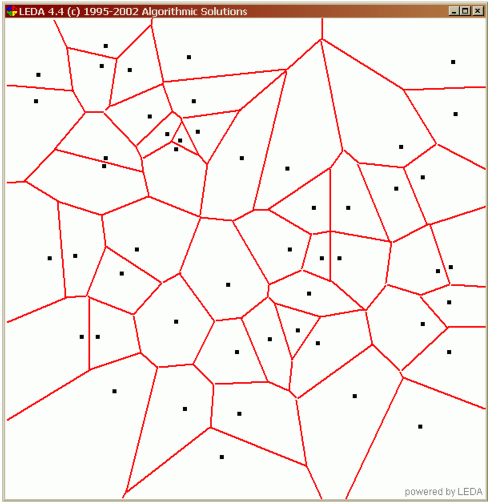
Voronoi Diagrams (chapter 7)

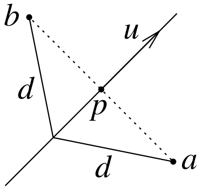
Elisha Sacks

Voronoi Diagram



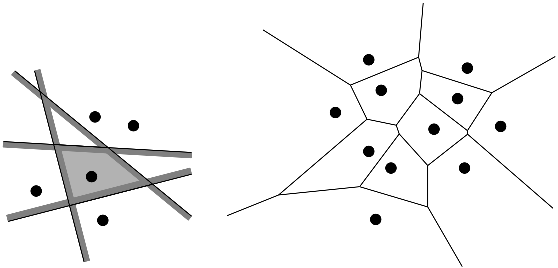
The Voronoi diagram of sites $p_1, \ldots, p_n$ is the subdivision in which each p_i has a cell $V(p_i)$ containing the points that are closer to it than to any other site.

Bisector



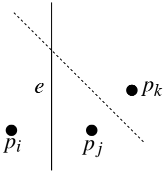
- Voronoi diagrams are defined in terms of bisectors.
- The bisector of a and b is the line through the midpoint $p = (a + b)/2$ of ab and with direction $u = (b_y - a_y, a_x - b_x)$ perpendicular to $b - a$.
- The points on the bisector are equidistant from a and b.
- The points on the a side are closer to a than to b.
- They form the open half space $h(a, b)$.

Structure



- The cell $V(p_i)$ is $\cap_{p_j \neq p_i} h(p_i, p_j)$, hence is convex.
- An edge e_{ij} consists of the points that are equidistant from p_i and p_j and are farther from every other site.
- A vertex v_{ijk} is the intersection point of e_{ij}, e_{jk}, e_{ik}. It is equidistant from p_i, p_j, p_k and is not closer to any other site.
- Four edges can meet at a vertex, but this is degenerate.

Connectedness

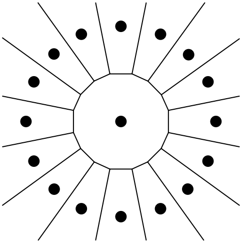


Theorem 7.2 The Voronoi diagram is connected unless the points are collinear in which case it consists of parallel lines.

Proof

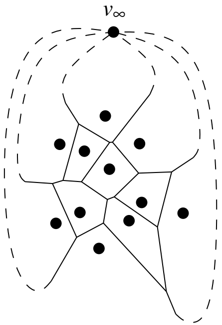
- If the Voronoi diagram were disconnected, it would have a cell bounded by two parallel bisectors, since cells are convex.
- Suppose $e = e_{ij}$ is such an edge.
- Let p_k be a site that is not collinear with p_i and p_j.
- The bisector of p_j and p_k intersects e.
- The part of e in $h(p_k, p_j)$ is closer to p_k than to p_j.
- This contradicts e being a Voronoi edge.
- The second part of the proof is easy.

Cell Complexity



- The complexity of a cell can be linear in the number of sites.
- The sites need not lie on a circle.

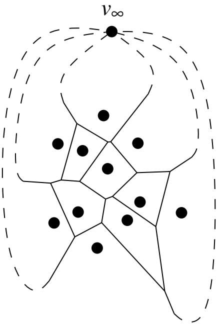
Voronoi Diagram Complexity



Theorem 7.3 The Voronoi diagram of n sites has at most $2n - 5$ vertices and $3n - 6$ edges.

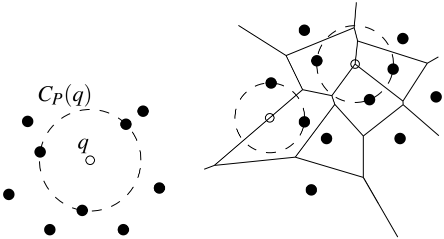
Proof We can assume $n \geq 3$ because the bounds are negative otherwise. Add a vertex v_∞ above the others. Deform and shorten the unbounded edges to make them incident on this vertex without intersecting each other. Wrap the edges that point left clockwise and the ones that point right counterclockwise.

Voronoi Diagram Complexity (continued)



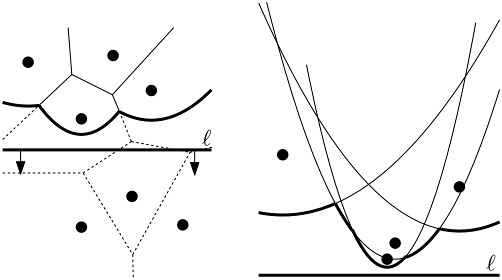
- Let n_v and n_e be the numbers of Voronoi vertices and edges.
- Every vertex is incident on at least three edges (why v_∞?) and every edge is incident on two vertices, so $2n_e \geq 3(n_v + 1)$.
- The Euler equation yields $2 = n_v + 1 - n_e + n \leq 2n_e/3 - n_e + n$, so $n_e \leq 3n - 6$.
- The Euler equation then yields $n_v = n_e + 1 - n \leq 3n - 6 + 1 - n = 2n - 5$.

Largest Empty Circle



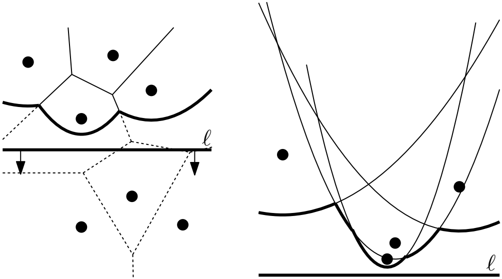
- The largest empty circle $C_P(q)$ is the largest circle centered at q whose interior contains no points of P.
- A point q is a Voronoi vertex iff the boundary of $C_P(q)$ contains 3 or more sites.
- The bisector of p_i and p_j defines a Voronoi edge iff it has a point q such that the boundary of $C_P(q)$ contains p_i and p_j, but no other sites.

Fortune's Algorithm



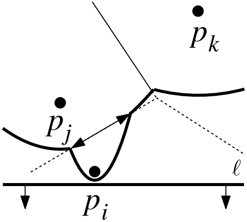
- Sweep a horizontal line ℓ downwards.
- The points equidistant to a site p_i and ℓ form a parabola.
- The lower envelope of the parabolas, called the beach line, consists of arcs whose meeting points are called breakpoints.
- The portion of the Voronoi diagram above the beach line is independent of the sites below ℓ.

Fortune's Algorithm (continued)



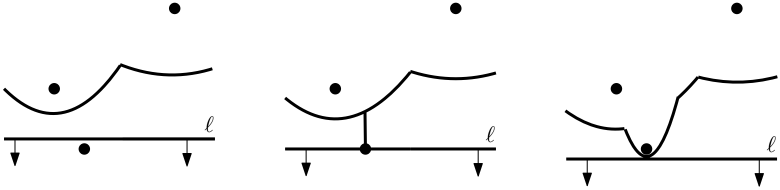
- The algorithm tracks the evolution of the beach line.
- The breakpoints trace the Voronoi edges.
- They enter and exit the sweep at sites and at Voronoi vertices.
- Breakpoints that never exit define unbounded edges.

Evolving Breakpoints



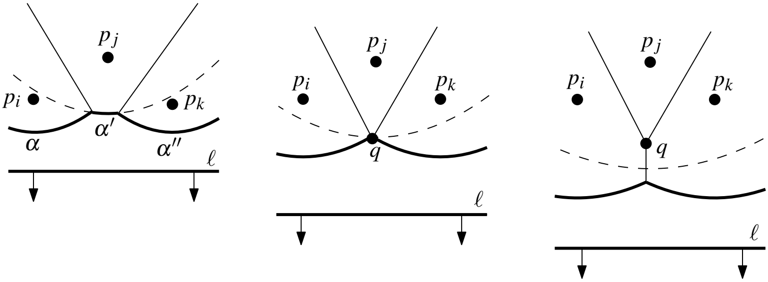
- Consider sites p_i and p_j.
- The breakpoints p_i/p_j and p_j/p_i trace the edge e_{ij}.
- The breakpoint p_i/p_j has p_i to its right and p_j to its left.
- Its tangent is 90° clockwise from $p_j - p_i$.

Site Event



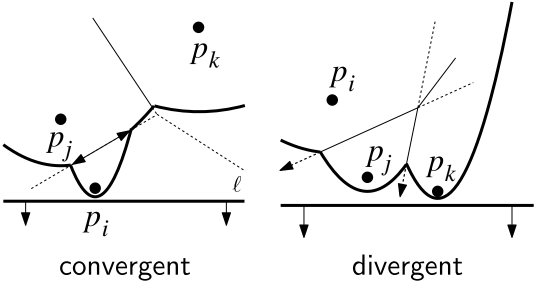
- The p_i site event occurs when ℓ crosses p_i.
- Split the arc above p_i at the x coordinate of p_i.
- Insert a p_i arc between the resulting two arcs.
- Insert $(p_j/p_i, p_i/p_j)$ into the beach line.
- p_j/p_i moves left and p_i/p_j moves right because $p_{iy} < p_{jy}$.

Circle Event



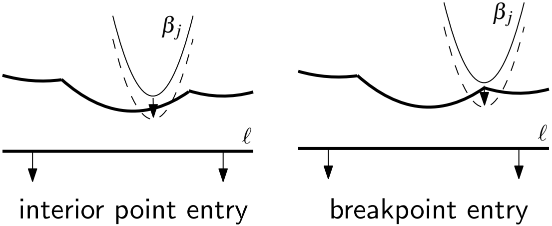
- A p_j circle event occurs when its arc α' leaves the beach line.
- The breakpoints p_i/p_j and p_j/p_k are replaced by p_i/p_k.
- The p_i arc α and the p_k arc α'' meet at p_i/p_k.
- The edges e_{ij}, e_{jk}, e_{ik} meet at $q = v_{ijk}$.
- The sweep line is tangent from below to $C_P(q)$.

Circle Event Detection



- Breakpoints p_i/p_j and p_j/p_k become adjacent.
- The center and radius of the p_i, p_j, p_k circle are q and r.
- If $\mathrm{LT}(p_i, p_j, p_k) < 0$, the breakpoints converge to q.
- A candidate circle event is posted at $q_y - r$.
- If $C_P(q)$ contains a site, p_i/p_j or p_j/p_k no longer exists at $q_y - r$, so the candidate is rejected.

No Other Events



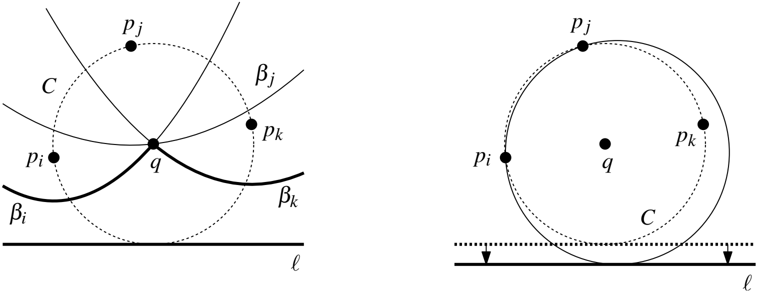
Lemma 7.6 A site event is the only way an arc enters the beach line.

Proof An arc might enter the beach line at an interior point of a prior arc or at a breakpoint of two prior arcs. We will rule both out.

No Interior Point Entry

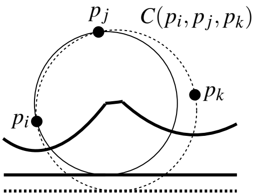
- Let β_i and β_j be the parabolas of p_i and p_j.
- Suppose β_i and β_j are tangent for some ℓ.
- $\beta_i(x) = \beta_j(x)$ and $\beta'_i(x) = \beta'_j(x)$
- $\beta_i(x) = \dfrac{(x - x_i)^2}{2(y_i - l)} + \dfrac{y_i + l}{2}$
- $\beta'_i(x) = \dfrac{x - x_i}{y_i - l}$
- $\beta'_i(x) = \beta'_j(x)$ implies $x - x_j = \dfrac{x - x_i}{y_i - l}(y_j - l)$
- $\beta_i(x) = \beta_j(x)$ implies $\dfrac{(x - x_i)^2}{(y_i - l)^2}(y_i - y_j) = y_j - y_i$
- This implies $y_i = y_j$, so $\beta'_i(x) = \beta'_j(x)$ implies $x_i = x_j$.

No Breakpoint Entry



- Suppose q is an entry point of β_j on the β_i/β_k breakpoint.
- There is a circle C clockwise through p_i, p_j, p_k tangent to ℓ.
- One vertex, say p_k, is closer than p_j to ℓ.
- This remains true as ℓ moves down (solid line and circle).
- The vertex p_k is closer than p_j to ℓ.

All Vertices Found



Lemma 7.8 Every Voronoi vertex is found by a circle event.

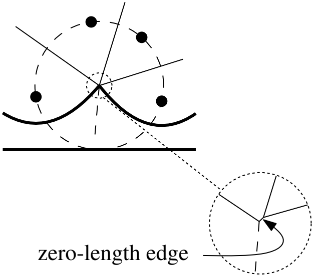
Proof Let sites p_i, p_j, and p_k be in clockwise order around the empty circle $C(p_i, p_j, p_k)$. Assume that no other sites are on C; the degenerate case is trickier.
Just before the sweepline reaches the bottom of C, it is tangent to an empty circle through p_i and p_j. The center of this circle is a p_i/p_j breakpoint. A p_j/p_k breakpoint exists by the same reasoning. These breakpoints are adjacent, so the circle C event occurs.

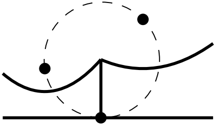
Algorithm Complexity

The complexity of Fortune's algorithm is $O(n \log n)$.

- There are $O(n)$ events by Theorem 7.3.
- There are $O(1)$ rejected circle candidates per event.
- Rejected candidates do not generate events or candidates.
- Handling an event is $O(\log n)$ for beach line updating.

Degenerate Cases





- The algorithm extends to the degenerate cases.
- Degenerate vertices lead to zero-length edges.
- Sites with equal y coordinates are usually harmless.
- Special logic is needed if they are the highest sites.
- A site below a breakpoint also leads to zero-length edges.

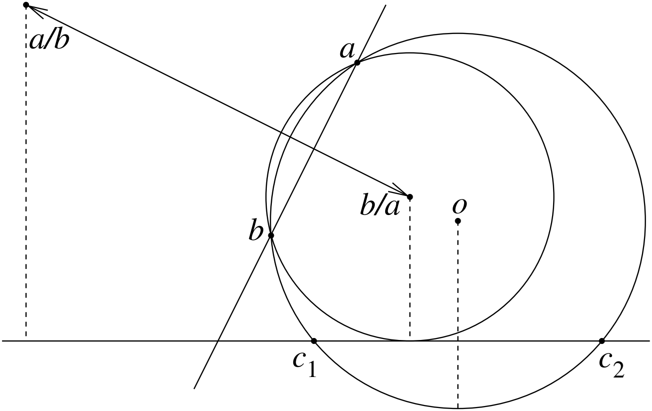
Implementation

1. Construct the subdivision vertices and edges.
   - Each breakpoint moves along an edge from tail to head.
   - Site event for p_i below p_j
     - Create p_i/p_j and p_j/p_i edges with null tails.
   - Circle event for p_i/p_j and p_j/p_k
     - Create the vertex v_{ijk},
     - Set the tails of p_j/p_i and p_k/p_j to v_{ijk},
     - Create the p_i/p_k edge with tail v_{ijk}, and
     - Create the p_k/p_i edge with a null tail.
2. Construct a bounding box for the vertices and the sites.
3. End the unbounded rays at the bounding box and refine it.
4. Construct the subdivision faces.

Beach Line

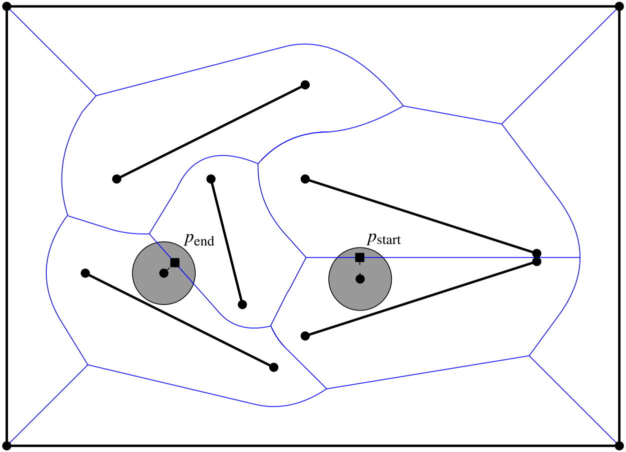
- The beach line is represented with a binary tree.
- The breakpoints are the internal nodes.
- The arcs are the leaf nodes.
- A p_i site event replaces a leaf with two internal nodes and three leafs.
- The only comparison is between p_{ix} and a breakpoint x.
- A circle event replaces two internal nodes and three leafs with one internal node and two leafs.
- No comparisons are performed.

Beach Line Order



- The book does not say how to compute the x order of a site c with respect to a breakpoint a/b or b/a.
- If $\mathrm{LT}(a, b, c) > 0$, c and a/b project to opposite sides ab.
- For b/a, let o be the center of the a, b, c circle.
- c projects to the same side of b/a and o (cases c_1 and c_2).

Voronoi Diagram of Line Segments



- Line segments cannot share endpoints.
- Bisectors consist of line segments and parabolic arcs.
- Fortune's algorithm generalizes; the book omits the details.