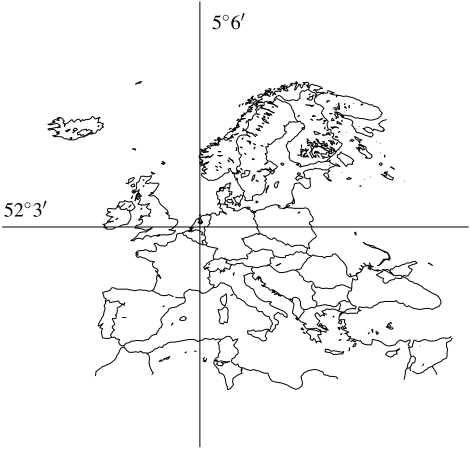
Point Location (chapter 6)

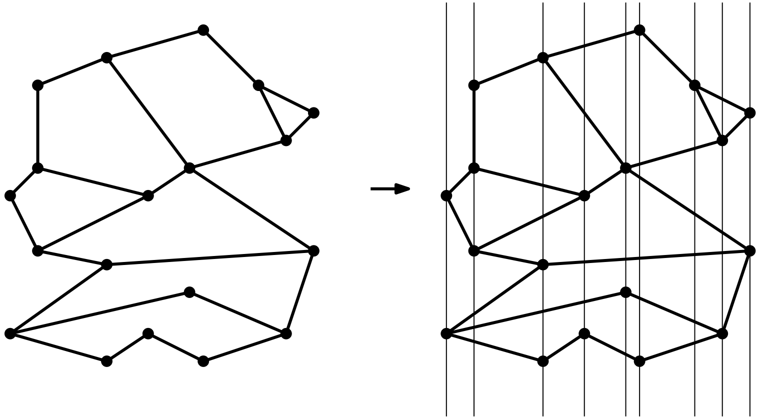
Elisha Sacks

Point Location



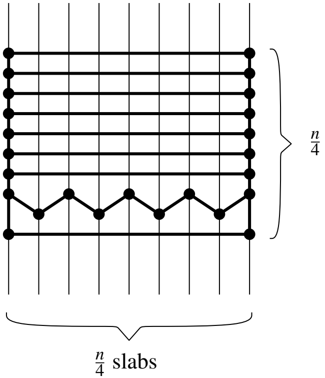
Find the face of a subdivision that contains a query point.

Slab Decomposition



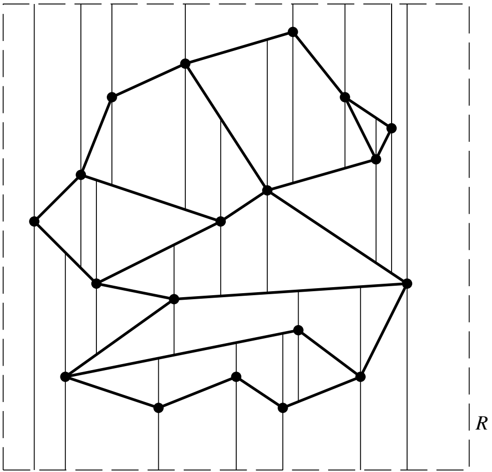
- Form slabs with verticals through the subdivision vertices.
- Sort the slabs along the x axis.
- Sort the edges of each slab along the y axis.
- Locate a point via two binary searches.

Complexity



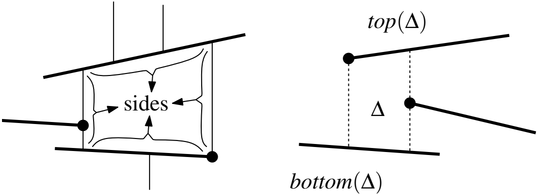
- Good point location time: $O(\log n)$.
- Bad space complexity: $O(n^2)$.
- Bad preprocessing time: $O(n^2 \log n)$.

Trapezoidal Map



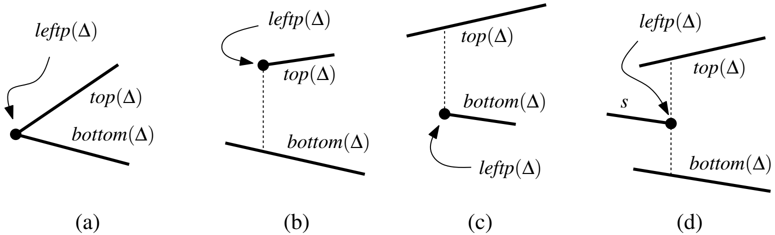
Overlay the subdivision with a box R and subdivide the bounded faces with verticals from each vertex to the edges above and below.

Trapezoidal Map Structure



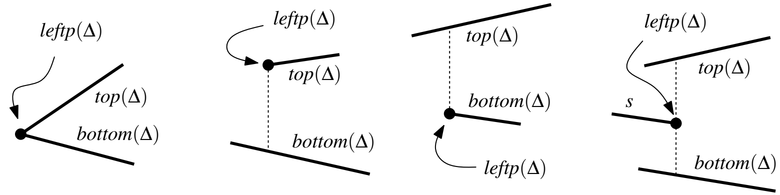
- The faces of the trapezoidal map are trapezoids and triangles.
- The top and bottom sides are segments of overlay edges.
- The left and right sides are verticals through overlay vertices.
- The left or right side of a triangle is a subdivision vertex.
- The straightforward proof appears in the textbook.
- We assume that subdivision vertex x coordinates are distinct.
- This assumption will be removed later.

Trapezoid Representation



- A trapezoid Δ is represented by its $top(\Delta)$ and $bottom(\Delta)$ edges and by its $leftp(\Delta)$ and $rightp(\Delta)$ vertices.
- The figure shows the cases for $leftp(\Delta)$.
- The cases for $rightp(\Delta)$ are analogous.
- The non-subdivision vertices are represented implicitly.

Space Complexity



(a) (b) (c) (d)

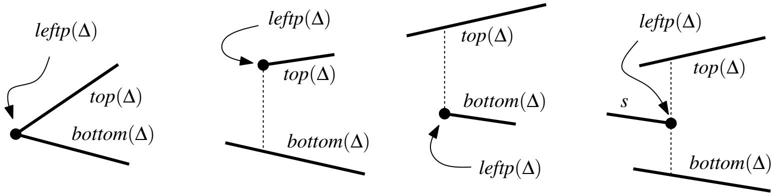
Lemma 6.2 A trapezoidal map of a subdivision with n edges has at most $6n + 4$ vertices and $3n + 1$ trapezoids.

Proof

Vertices: 4 from R, $2n$ from the subdivision, and $2 \times 2n$ from the vertical sides.

Trapezoids: Every trapezoid has a *leftp*. The bottom of R is the *leftp* of one trapezoid, a subdivision edge defines at most two *leftp* with its left endpoint and at most one *leftp* with its right endpoint.

Space Complexity



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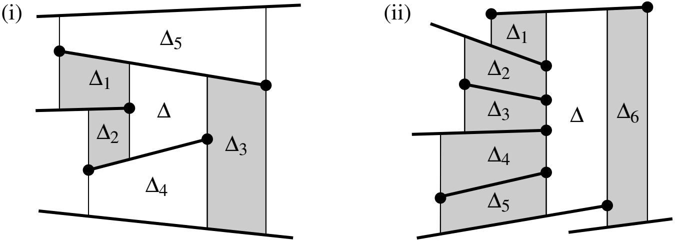
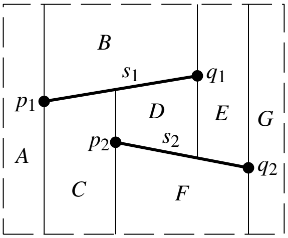
Why doesn't leftp(Δ) *define three trapezoids in (a)?*

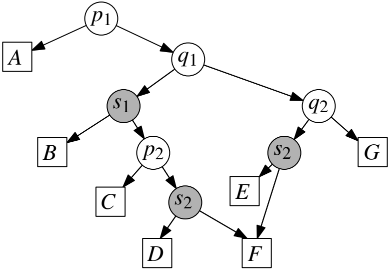
Neighbors



- Distinct x coordinates imply that a trapezoid has at most two left neighbors and at most two right neighbors (i).
- The left neighbors are encoded by top, bottom, and leftp.
- The right neighbors are encoded by top, bottom, and rightp.
- Each trapezoid stores pointers to its neighbors.
- Duplicate x coordinates allow any number of neighbors (ii).

Trapezoidal Map and Search Graph



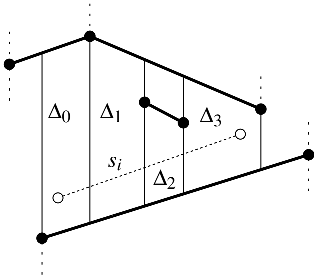


- Find the trapezoid containing a point a by traversing a graph.
- A point node u (white) branches left if $a_x < u_x$.
- An edge node s (grey) branches left if a is above s.
- The leafs point to trapezoids.

Incremental Construction of the Trapezoidal Map

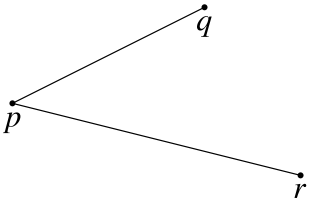
1. Create a search graph for the bounding box R.
2. Process the edges s_i of the subdivision in random order.
   2.1 Find the trapezoids that s_i intersects.
   2.2 Update the trapezoids.
   2.3 Update the search graph.

Finding the Trapezoids



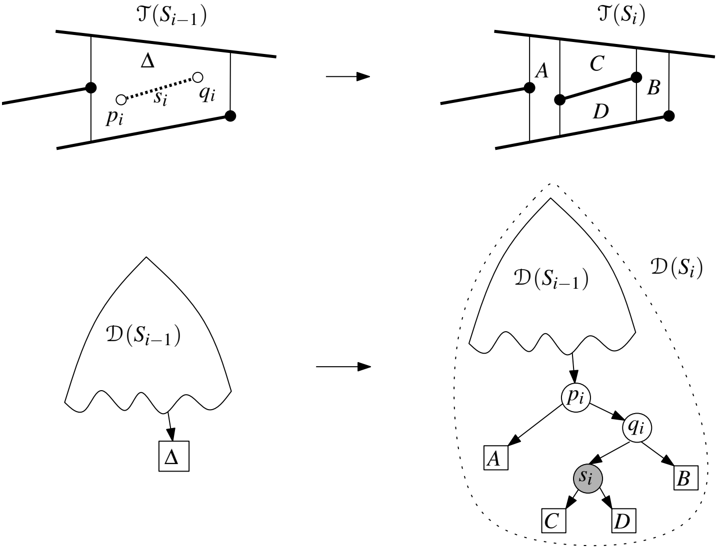
1. Find the trapezoid Δ_0 that contains the left endpoint using the search graph.
2. Move right: Δ_i is the upper/lower right neighbor of Δ_{i-1} if s_i is above/below $rightp(\Delta_{i-1})$.
3. Stop at the trapezoid Δ_k that contains the right endpoint.

Identities in Graph Construction

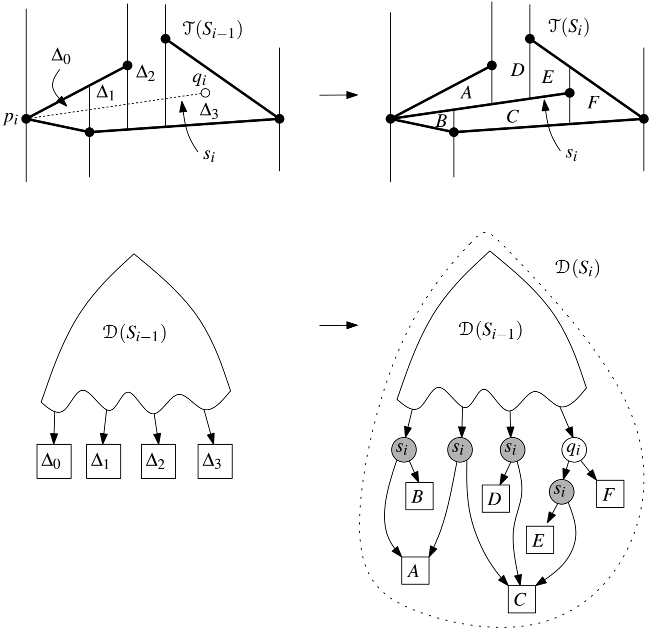


- Suppose pr is inserted when pq is in the graph.
- The p_x test and the pq test are identities.
- These cases are handled by secondary tests.
  - The p_x test branches right.
  - The pq test branches right when $\mathrm{LT}(q, p, r) > 0$.

Update: One Trapezoid



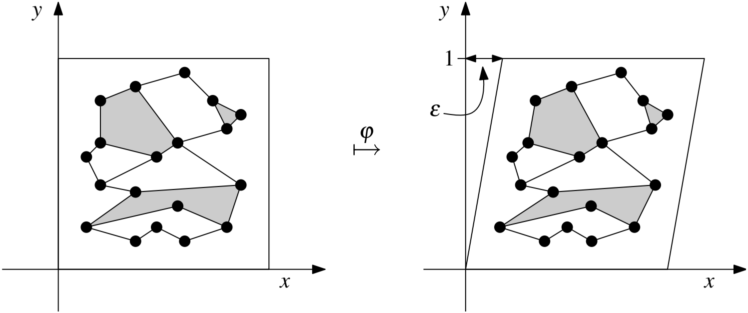
Update: General Case



Degeneracy

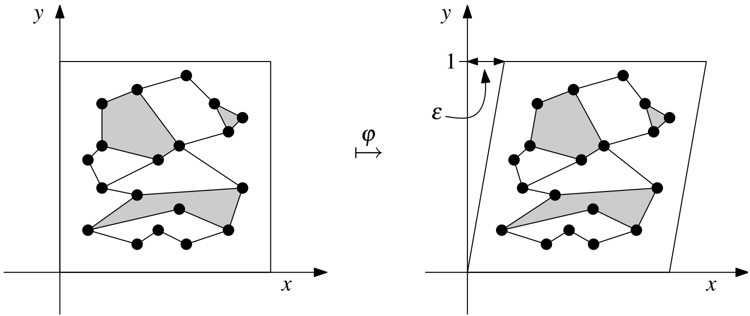
- A query point on an edge reports the edge.
- A query point equal to a vertex reports the vertex.
- Equal x coordinates can be eliminated by input perturbation or by symbolic perturbation.

Symbolic Perturbation



- Shear the vertices: $(x, y) \to (x + \epsilon y, y)$.
- The x order is preserved for small enough positive ϵ.
- Predicates are evaluated *without* computing ϵ or shearing.
    - The x order is replaced by lexicographic order.
    - The point/edge predicate is unchanged.

Symbolic Perturbation



- Shear the vertices: $(x, y) \to (x + \epsilon y, y)$.
- The x order is preserved for small enough positive ϵ.
- Predicates are evaluated *without* computing ϵ or shearing.
  - The x order is replaced by lexicographic order.
  - The point/edge predicate is unchanged.
- Does this strategy apply to algorithms that construct vertices?

Randomized Expected Complexity

Theorem 6.3 The trapezoidal map of n segments has $O(\log n)$ query time, $O(n)$ space complexity, and $O(n \log n)$ construction time in randomized expectation.

Proof We will prove each bound in turn.

Expected Query Time

- The segments are inserted in random order $s_1, \ldots, s_n$.
- The first i segments are $S_i = \{s_1, \ldots, s_i\}$.
- Let P_i be the probability that inserting s_i creates nodes on the path through the search graph to the trapezoid that contains q.
- Let $\Delta_q(S_i)$ denote the trapezoid of $\mathcal{T}(S_i)$ that contains q.
- Key fact: $P_i = Pr[\Delta_q(S_i) \neq \Delta_q(S_{i-1})]$.
- If s_i is removed, $\Delta_q(S_i)$ vanishes with probability $\leq 4/i$.
  - s_i equals its top or bottom.
  - s_i is the only segment incident on its leftp or rightp.
- Let X_i nodes on the path to q be created when s_i is inserted.
- Case analysis shows that $X_i \leq 3$, so $E[X_i] \leq 3P_i$.
- The query time for q is linear in the sum $X = X_1 + \cdots + X_n$.
- Using linearity of expectation

$$E[X] = \sum_{i=1}^{n} E[X_i] \leq \sum_{i=1}^{n} 3P_i \leq \sum_{i=1}^{n} \frac{12}{i} = 12H_n < 12(\log n + 1)$$

Expected Space Complexity

- The space complexity is proportional to the number of nodes.
- Let k_i be the number of trapezoids created by inserting s_i.
- The graph grows by k_i leafs and $k_i - 1$ internal nodes.
- The number of nodes is bounded by $2(k_1 + \cdots + k_n)$.
- Define $\delta(t, s)$ to equal 1 if t vanishes from $\mathcal{T}(S_i)$ when s is removed and to equal 0 otherwise.
- Average k_i over the i choices of s_i within S_i.

$$
\begin{aligned}
E[k_i] &= \frac{1}{i} \sum_{s \in S_i} \sum_{t \in \mathcal{T}(S_i)} \delta(t, s) = \frac{1}{i} \sum_{t \in \mathcal{T}(S_i)} \sum_{s \in S_i} \delta(t, s) \\
&\leq \frac{1}{i} \sum_{t \in \mathcal{T}(S_i)} 4 = \frac{4|\mathcal{T}(S_i)|}{i} \leq \frac{4(3i+1)}{i} < 13
\end{aligned}
$$

- The space complexity is $2 \sum_{i=1}^{n} E[k_i] = O(n)$.

Expected Construction Time

- Time for s_i is $O(\log i)$ lookup of left endpoint plus $E[k_i] = 1$.
- Construction time is $\sum_{i=1}^{n}(\log i + 1) = n \log n$.