# Projective Geometry

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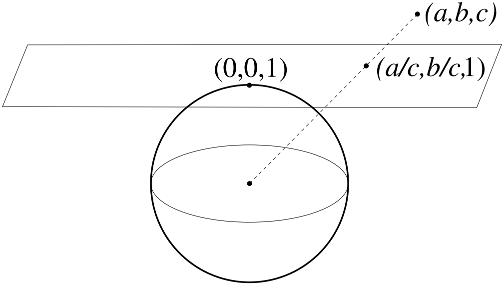
# Motivation

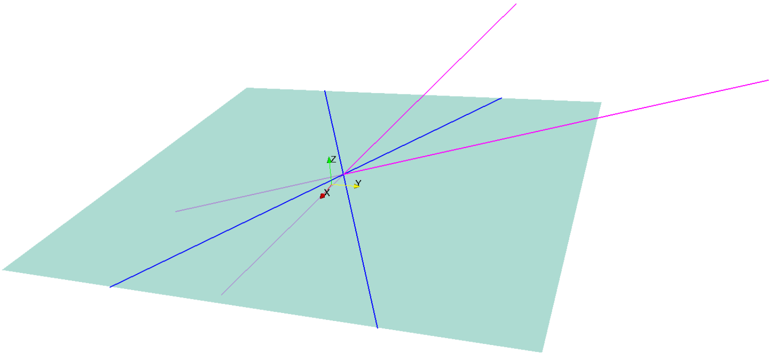
- The projective plane adds points at infinity to the affine plane.
- Two parallel lines intersect at a point at infinity.
- Asymptotes of algebraic curves are points at infinity.
- These concepts remove special cases from affine geometry.
- Any two projective lines intersect at a unique point.
- Every projective algebraic curve consists of closed loops.

# Projective Points



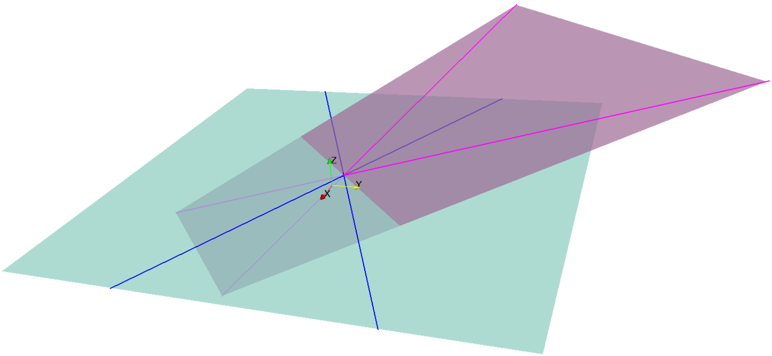

- A projective point is a line through the origin of $\Re^3$.
- Its homogenous coordinates are any point $(a, b, c)$ on the line.
- If $c \neq 0$, it intersects the $z = 1$ plane at $(a/b, b/c, 1)$ and represents the affine point $(a/c, b/c)$.
- If $c = 0$, it is at infinity.

# Projective Points



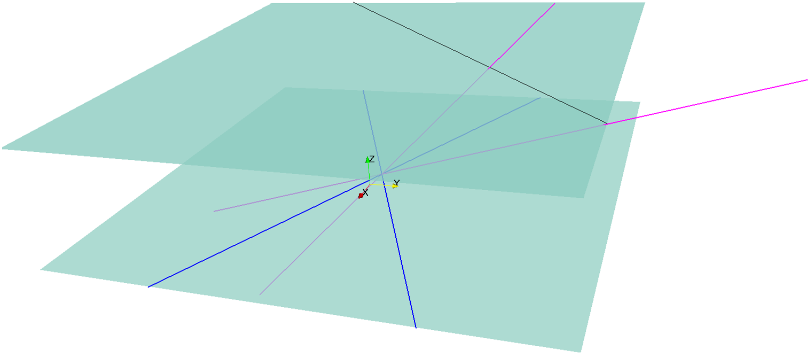

- The pink lines are affine points.
- The blue lines are points at infinity.

# Projective Lines



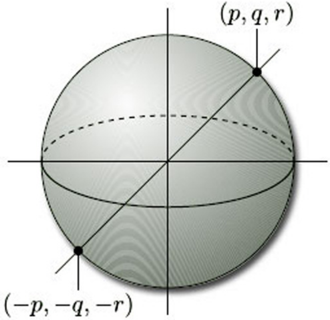

- A projective line is a plane through the origin of $\Re^3$.
- The line $ux + vy + wz = 0$ is written as $\langle u, v, w \rangle$.
- It consists of the affine points $(a, b, 1)$ with $(a, b)$ on the affine line $ux + vy + w = 0$, plus $(-v, u, 0)$ at infinity.
- The line at infinity $z = 0$ consists of all the points at infinity.

# Plane Model



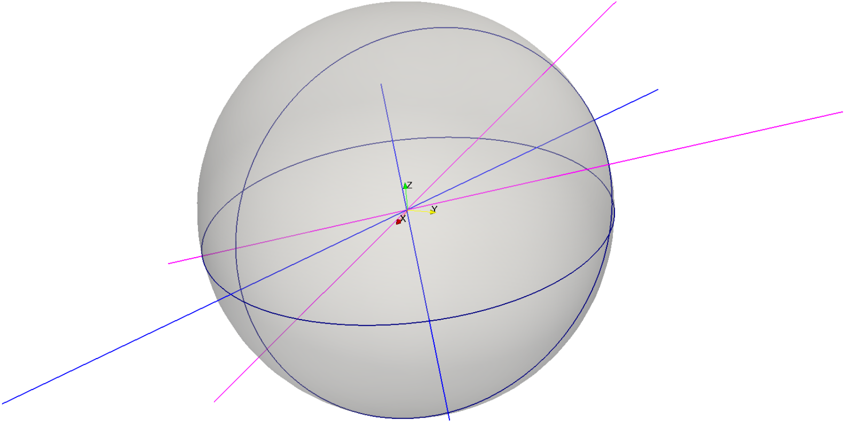

- Map an affine point to its intersection with the $z = 1$ plane.
- Map the line at infinity to the $z = 0$ plane.
- Affine lines are on the $z = 1$ plane.
- Their points at infinity are on the $z = 0$ plane.

# Sphere Model



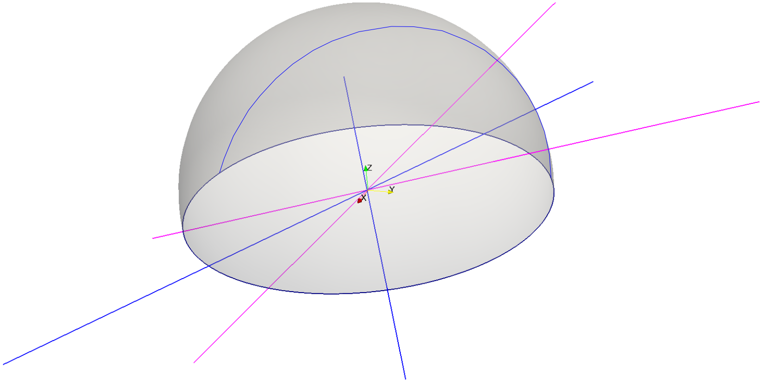

- Map a point to its two intersections with the unit sphere.
- Lines map to great circles.
- The line at infinity maps to the equator.

# Sphere Model




- The pink lines (affine points) lie on the great circle.
- The blue lines (points at infinity) lie on the equator.

# Hemisphere Model




- Map a point to its intersection with the northern hemisphere.
- Affine lines map to great semicircles.
- The line at infinity maps to the equator.

# Points and Lines

- The line through points $p$ and $q$ has normal $p \times q$.
- Lines $m$ and $n$ intersect at the point $p = m \times n$.
- If $m$ and $n$ are affine and non parallel, $p$ is affine.
- If $m$ and $n$ are parallel, $p$ is at infinity.
  $(u, v, w) \times (u, v, w') = (-v(w - w'), u(w - w'), 0) = (-v, u, 0)$
- If $m$ is the line at infinity, $p$ is $n$'s point at infinity.
  $(0, 0, 1) \times (u, v, w) = (-v, u, 0)$
- The line at infinity is parallel to every affine line.

# Examples

The affine lines $x + y - 1 = 0$ and $x - y - 1 = 0$ intersect at $(1, 0)$. The projective lines $x + y - z = 0$ and $x - y - z = 0$ intersect at $(1, 1, -1) \times (1, -1, -1) = (1, 0, 1)$.

The affine lines $x - y - 1 = 0$ and $x - y - 2 = 0$ are parallel. The projective lines $x - y - z = 0$ and $x - y - 2z = 0$ intersect at $(1, -1, -1) \times (1, -1, -2) = (1, 1, 0)$.

The affine points $(1, 1)$ and $(2, 3)$ define the line $-2x + y + 1 = 0$. The projective points $(1, 1, 1)$ and $(2, 3, 1)$ define the line $-2x + y + z = 0$, since $(1, 1, 1) \times (2, 3, 1) = (-2, 1, 1)$.

The affine line through $(a, b)$ in direction $(c, d)$ is the projective line $(a, b, 1) \times (c, d, 0)$.

# Duality

There is a natural duality between the point $p = (a, b, c)$ and the line $\hat{p} = \langle a, b, c \rangle$.

Unlike the affine case, every line has a dual.

If a point $p$ is on a line $l$, $\hat{l}$ is on $\hat{p}$, since the original equation is $p \cdot l = 0$ and the dual equation is $\hat{l} \cdot \hat{p} = 0$.

If a line $l$ passes through points $p$ and $q$, $\hat{p}$ and $\hat{q}$ intersect at $\hat{l}$, since $l = p \times q$ implies $l \cdot p = 0$ and $l \cdot q = 0$, so $\hat{l} \cdot \hat{p} = 0$ and $\hat{l} \cdot \hat{q} = 0$.

# Projective Varieties

A projective variety is the zero set of a homogeneous polynomial $p(x, y, z)$; every term of the polynomial has the same degree $d$.

Examples: a projective line is homogeneous with $d = 1$ and $xy - z^2$ is homogeneous with $d = 2$.

A homogeneous polynomial is zero or nonzero for all the homogeneous coordinates of a projective point.

The projective variety $p(x, y, z) = 0$ consists of the affine variety $p(x, y, 1) = 0$, which is its intersection with the plane $z = 1$, plus the points at infinity $p(x, y, 0) = 0$, which are its intersection with the plane $z = 0$.

Example: $xy - z^2 = 0$ consists of the hyperbola $xy = 1$ plus the points at infinity $(1, 0, 0)$ and $(0, 1, 0)$.

# Homogenization

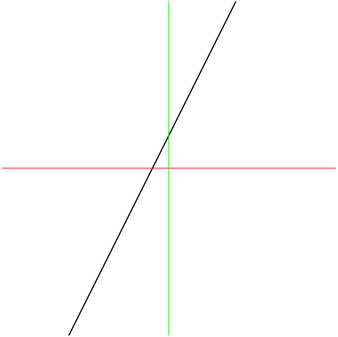
*Homogenization:* Convert an affine polynomial $p(x, y) = 0$ to a homogeneous polynomial in $x, y, z$ by substituting $x/z$ for $x$ and $y/z$ for $y$ then clearing the denominator.

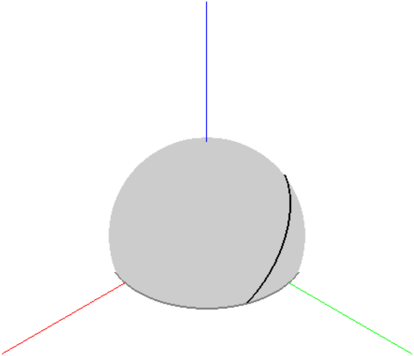
Example: the hyperbola $xy - 1 = 0$ homogenizes to $xy - z^2 = 0$.

*Dehomogenization:* Convert a homogeneous polynomial to an affine polynomial by substituting $z = 1$.

Let $q(x, y, z)$ be the homogenization of $p(x, y)$. The affine variety of $p$ equals the affine part of the projective variety of $q$, that is the points with $z = 1$. The points at infinity of $q$ are the zeroes of the leading (highest degree) terms of $p$, since the other terms of $q$ are zero for $z = 0$.

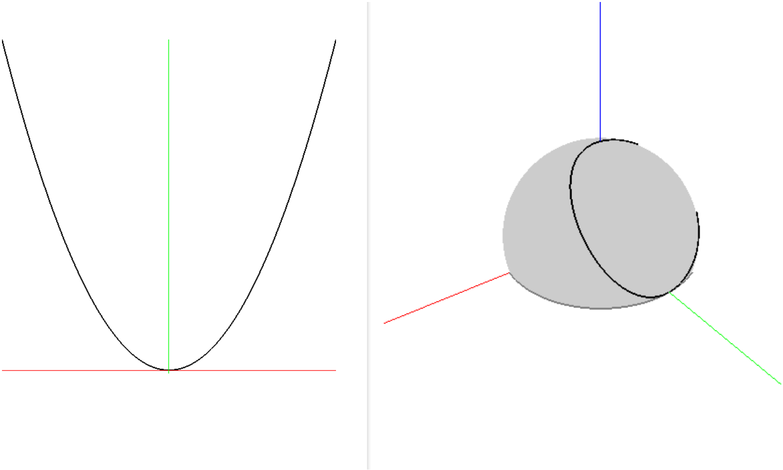
# Line





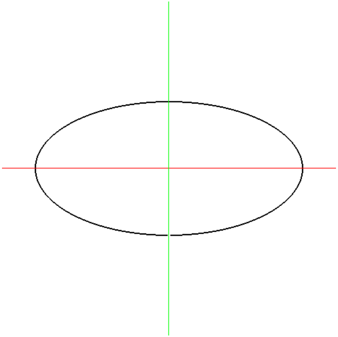
The line $y = 2x + 2$ homogenizes to $2x - y + 2z = 0$ with point at infinity $(1, 2, 0)$ that equals $(0.447, 0.894, 0)$ in the hemisphere model. This point converts the affine line into a loop.

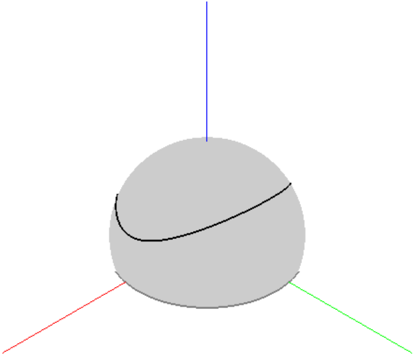
# Parabola



The parabola $y = x^2$ homogenizes to $yz - x^2 = 0$ with point at infinity $(0, 1, 0)$ that converts the affine parabola into a loop.

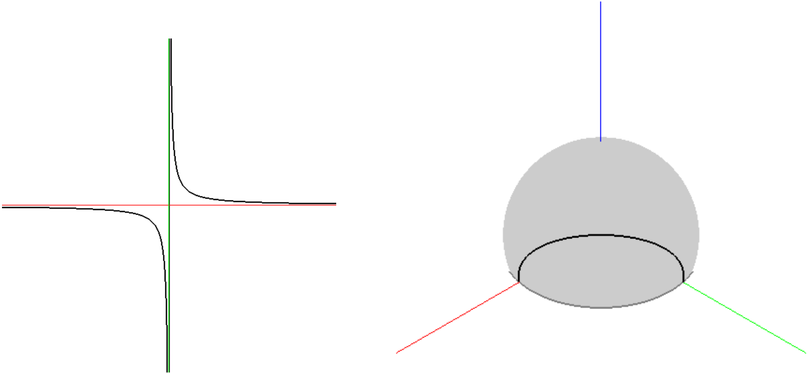
# Ellipse





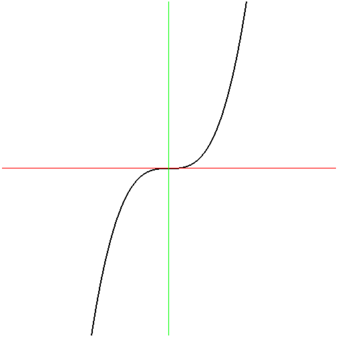
The ellipse $x^2 + 4y^2 = 4$ homogenizes to $x^2 + 4y^2 - 4z^2 = 0$ with no points at infinity, since the affine ellipse is already closed.

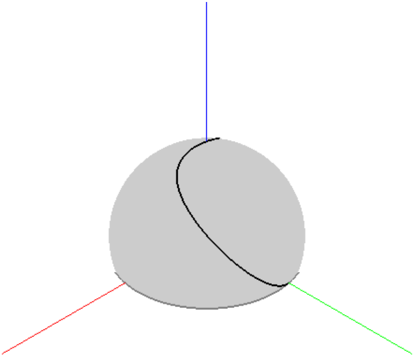
# Hyperbola



The hyperbola $xy = 1$ homogenizes to $xy - z^2 = 0$ with points at infinity $(1, 0, 0)$ and $(0, 1, 0)$. These points convert the two components of the affine hyperbola into a single loop.

# Cubic





The cubic $y = x^3$ homogenizes to $yz^2 - x^3 = 0$ with point at infinity $(0, 1, 0)$ that converts the affine variety to a loop.

# Complex Projective Geometry

The true setting for algebraic geometry is complex projective space.

Example: The circle $x^2 + y^2 = 1$ homogenizes to $x^2 + y^2 = z^2$ with points at infinity $(\pm 1, i)$.

**Bezout's theorem** If polynomials $p$ and $q$ of degrees $m$ and $n$ do not have a common component, they have $mn$ complex projective roots counting multiplicity.

Example: The intersection of two circles consists of two real or complex affine points and the two points at infinity $(\pm 1, i)$.

# Projective Geometry in $n$ Dimensions

- Every affine space $k^n$ has a projective space $P(k^n)$.
- The projective points are lines through the origin of $k^{n+1}$.
- The homogeneous coordinates are $(x_1, \ldots, x_{n+1})$.
- If $x_{n+1} \neq 0$, $x$ maps to the affine point $(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}})$.
- If $x_{n+1} = 0$, $x$ is at infinity.
- The plane, sphere, and hemisphere models generalize.
- The points at infinity are isomorphic to $P(k^{n-1})$.
- The space $P(\Re^3)$ is used in graphics.

# Limitations of Projective Geometry

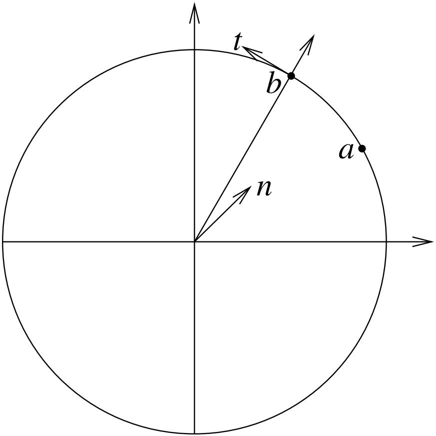
- Although the projective plane eliminates the special cases of the affine plane, it also has disadvantages.
- The projective plane is not orientable.
- Lines have one side: removing a line leaves a connected set.
- Segments are ambiguous: two points split their line into two connected parts that cannot be distinguished.
- Likewise, the direction from $a$ to $b$ is ambiguous, e.g. each point at infinity lies in two directions from every affine point.
- Convexity is undefined.

# Oriented Projective Geometry

- Stolfi [1] defines an oriented version of projective geometry that solves these problems at the cost of increased complexity.
- Each projective point is split into two oriented points: the line $ka$ is split into the rays $ka$ and $-ka$ with $k > 0$.
- Each projective line is split into two oriented lines likewise.
- In the sphere model, opposite points are no longer identified and great circles are oriented.
- The convex hull of a set of points is the dual of the envelope of the dual lines.

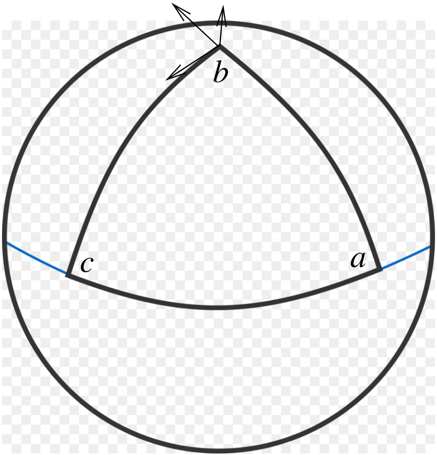
[1] J. Stolfi, Oriented Projective Geometry, Academic Press, 1991.

# Spherical Computational Geometry




- A point $a$ has normal vector $a$.
- A segment $ab$ lies in the plane with normal $n = a \times b$ and is traversed counterclockwise around $n$.
- The tangent to $ab$ at $b$ is $t(ab, b) = (a \times b) \times b = n \times b$.

# Spherical Computational Geometry




- The path *abc* is a left turn if $b \cdot t(ab, b) \times t(bc, b) > 0$.
- The segment intersection predicate is as before.

# Spherical Computational Geometry

- Some algorithms transfer easily from the plane to the sphere.
- Some rely on properties of the plane that differ on the sphere.
- For example, the sum of the angles of a triangle is not $180^\circ$.
- Spherical geometry is an instance of Riemann geometry.