3D Convex Hulls (chapter 11)

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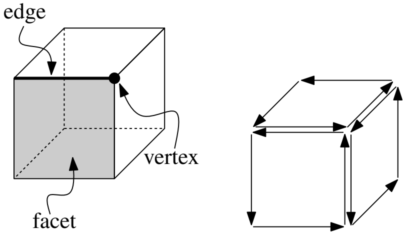
Convex Hull of 3D Points

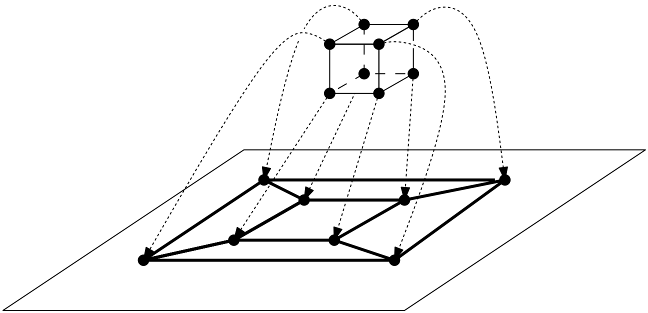
- Smallest convex set that contains the points
- Convex polyhedron
- Used in shape approximation and collision detection
- 2D Voronoi diagram and Delaunay triangulation (next class)

Boundary Representation



- A vertex has coordinates and incident edges.
- An edge has a tail, a twin, a next edge, and a facet.
- Edge loops bound facets.
- A facet has one edge per boundary loop.
- A convex polyhedron is a convex set bounded by convex facets such that every edge is incident on one facet.

Space Complexity



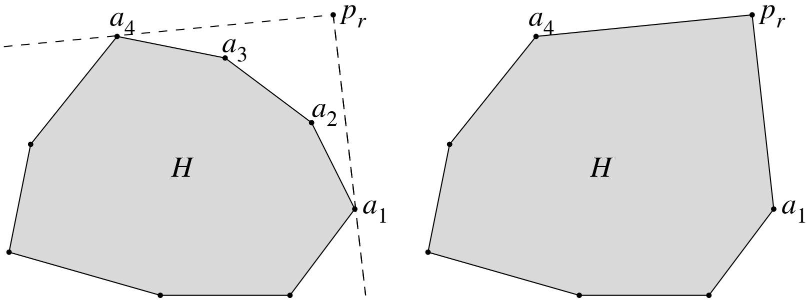
Theorem 11.1 A convex polyhedron with n vertices has at most $3n - 6$ edges and at most $2n - 4$ facets.

Proof Euler's formula for a genus zero polyhedron with e edges and f facets is $n - e + f = 2$. Every facet has at least three edges and every edge is incident on two facets, so $2e \geq 3f$.

$n + f - 2 = e$ implies $n + f - 2 \geq 3f/2$ hence $f \leq 2n - 4$.

$e = n + f - 2$ implies $e \leq n + 2n - 4 - 2 = 3n - 6$.

Incremental 2D Algorithm



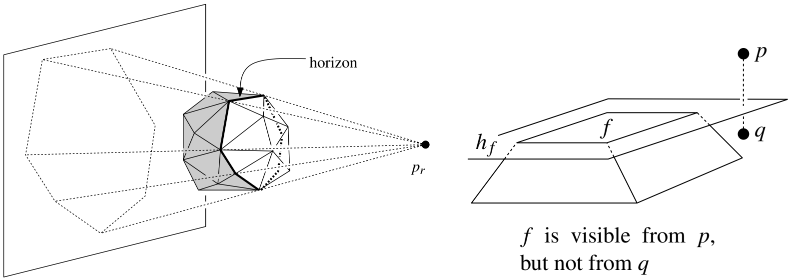
1. Randomize the points to $p_1, \ldots, p_n$.
2. Initialize the hull to $H = p_1 p_2 p_3$ in counterclockwise order.
3. For $r = 4$ to n:

   If p_r is outside of H

      Remove the visible edges $a_1 a_2, \ldots, a_{k-1} a_k$.

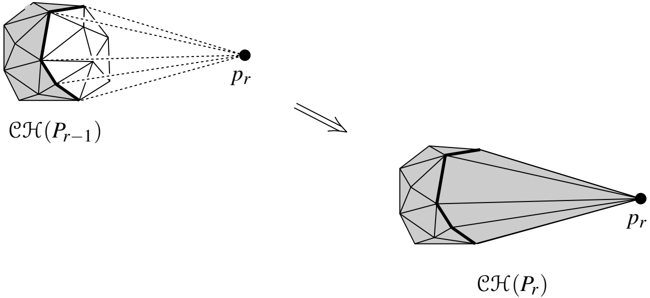
      Create edges $a_1 p_r$ and $p_r a_k$.

Incremental 3D Algorithm



- The same idea works in 3D.
- A facet is visible if p_r is in its positive half-space.
- The visible facets form a surface.
- The boundary of this surface is the *horizon* curve.

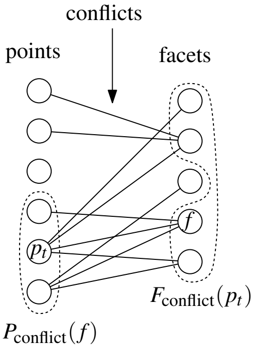
Incremental 3D Algorithm



1. Randomize the points to $p_1, \ldots, p_n$.
2. Initialize the hull to $\mathcal{CH}(P_4) = p_1 p_2 p_3 p_4$.
3. For $r = 5$ to n:
   If p_r is outside of $\mathcal{CH}(P_{r-1})$
      Remove the visible facets.
      Create facets that link p_r to the horizon edges.

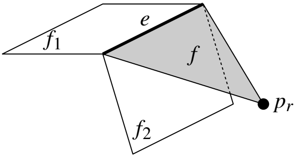
Note: need to list horizon edges counterclockwise around p_r.

Conflict Graph



- Each uninserted point is linked to its visible facets.
- Each facet is linked to its visible uninserted points.
- The graph is initialized with the facets of $\mathcal{CH}(P_4)$ and the uninserted points $p_5, \ldots, p_n$.
- It is updated during point insertion.
- The Delaunay triangulation algorithm uses the same idea.

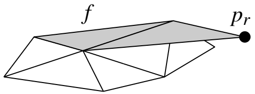
Conflict Graph Update



1. Remove the p_r node and its edges.
2. For each new horizon edge e with new facet f:
   - 2.1 Let S be the points that conflict with the old e facets f_1 and f_2.
   - 2.2 Remove the facet nodes and their edges.
   - 2.3 Create the f node.
   - 2.4 Add edges from the f node to its visible points in S.

Correctness proof: If a point can see f in $\mathcal{CH}(P_r)$, it can see e in $\mathcal{CH}(P_r)$, so it can see e in $\mathcal{CH}(P_{r-1}) \subset \mathcal{CH}(P_r)$, so it can see a facet incident on e in $\mathcal{CH}(P_{r-1})$.

Degenerate Cases



Degeneracy: A point p_r is coplanar with a facet f.
If f does not contain p_r, it is visible.
The new facet is not a triangle.
The new facet has the same conflicts as the old one.

Degeneracy: The points p_1, p_2, p_3, p_4 are coplanar.
Prevented by randomization. Or pick four other points.

Degeneracy: The points are coplanar.
use a 2D algorithm.

Degeneracy: The points are collinear.
Return a line segment.

Complexity

- We prove that the expected number of facets created is $O(n)$.
- Let s be the total number of points in the S sets in step 2.1.
- The expected time complexity is $O(n + s)$.
- The book proves that s is $O(n \log n)$.
- The proof is a complicated variant of earlier proofs.

Number of Facets

Lemma 11.3 The expected number of facets created is at most $6n - 20$.

Proof
$\mathcal{CH}(P_4)$ has four facets.
The number of facets created by p_r is the number of edges incident on p_r in $\mathcal{CH}(P_r)$.
There are at most $3r - 6$ edges each incident on two vertices.
The expected number of p_r facets is $(6r - 12)/r < 6$.
The sum over the n points is at most $4 + 6(n - 4) = 6n - 20$.