Geometry of Curves and Surfaces

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Planar Vector Geometry

- Vectors represent positions and directions.
- Vector u has Cartesian coordinates $u = (u_x, u_y)$.
- Inner product: $u \cdot v = u_x v_x + u_y v_y$.
- Projection of u onto v: $(u \cdot v / v \cdot v)v$.
- Vector length: $||u|| = \sqrt{u \cdot u}$.
- Unit vector: $u/||u||$.
- Cross product: $u \times v = u_x v_y - u_y v_x$
- Let α be the angle between u and v.
- $u \cdot v = ||u|| \cdot ||v|| \cdot \cos \alpha$.
- $u \times v = ||u|| \cdot ||v|| \cdot \sin \alpha$.

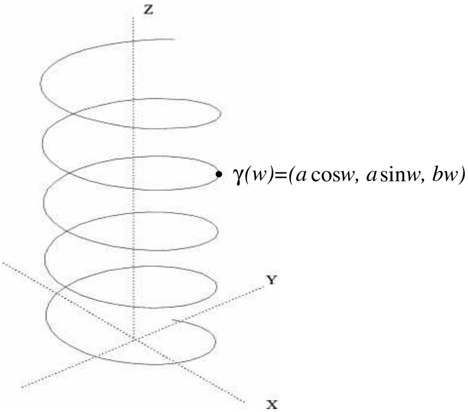
Spatial Vector Geometry

- Vectors represent positions and directions.
- Vector u has coordinates $u = (u_x, u_y, u_z)$.
- Inner product: $u \cdot v = u_x v_x + u_y v_y + u_z v_z$.
- Projection of u onto v: $(u \cdot v / v \cdot v) v$.
- Vector length: $||u|| = \sqrt{u \cdot u}$.
- Unit vector $u/||u||$.
- Cross product:
  $u \times v = (u_y v_z - u_z v_y, u_z v_x - u_x v_z, u_x v_y - u_y v_x)$
- Let α be the angle between u and v.
- $u \cdot v = ||u|| \cdot ||v|| \cdot \cos \alpha$.
- $u \times v = (||u|| \cdot ||v|| \cdot \sin \alpha)\, n$ with n a unit vector perpendicular to u and v.

Plane Curves

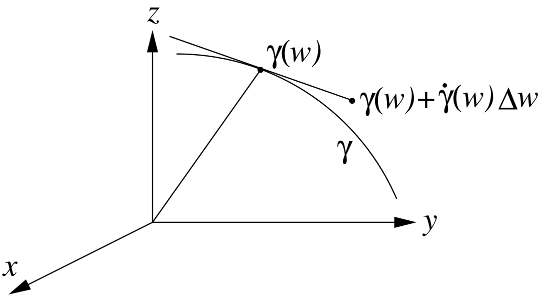
- Explicit: $y = f(x)$
- Implicit: $f(x, y) = 0$
- Parametric: $\gamma(w) = (x(w), y(w))$
- Example: circle with center o and radius r
  - explicit: $y = o_y \pm \sqrt{r^2 - o_x^2}$
  - implicit: $(x - o_x)^2 + (y - o_y)^2 = r^2$
  - parametric: $\gamma(w) = (o_x + r \cos w, o_y + r \sin w)$.

Space Curves



- Parameteric: $\gamma(w) = ((x(w), y(w), z(w)).$
- Most results apply to plane curves after dropping z.
- Implicit are rarely useful: $f(x, y, z) = 0, g(x, y, z) = 0.$

Tangent Vectors



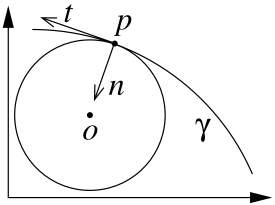
- The velocity vector at w is $\dot{\gamma}(w)$.
- The tangent line is $\gamma(w) + \dot{\gamma}(w)\Delta w$.
- The speed is $v = ||\dot{\gamma}||$.
- The length of the curve is the integral of v.

Length Parameterization

- Let $s(w) = \int_{u=0}^{w} v$ denote the length of the curve γ on $[0, w]$.
- The length parameterization of γ is $\gamma(s)$.
- The curve $\gamma(s)$ has unit speed, so its length on $[0, z]$ is z.
- Rewriting $\gamma(w)$ as $\gamma(s)$ is impractical.
- Changing variables at a point is easy using the chain rule.
- We use the notation $\gamma' = \dfrac{\partial \gamma}{\partial s}$.
- γ' is a unit vector.

  Proof: $\dot{\gamma} = \dfrac{\partial \gamma}{\partial s}\dfrac{\partial s}{\partial w} = \gamma' \dot{s} = \gamma' v$, so $\gamma' = \dfrac{\dot{\gamma}}{v} = \dfrac{\dot{\gamma}}{||\dot{\gamma}||}$
- The unit tangent is denoted $t = \gamma'$.

Curvature



The curvature of a curve γ at a point p measures its deviation from the tangent line at p.

- The curvature is $\kappa = ||t'||$.
- The principal normal is $n = t'/\kappa$.
- n is orthogonal to t: $t \cdot t = 1$ implies $(t \cdot t)' = 2t \cdot t' = 0$.
- A circle of radius r has constant curvature $1/r$.
- Two curves have second-order contact at a common point when they have the same unit tangent and curvature.
- Every curve has second-order contact at p with a circle of radius $r = 1/|\kappa|$ whose center is $o = p + rn$.

Curvature Computation

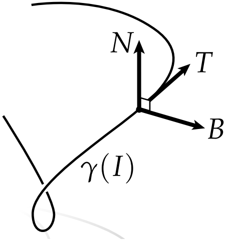
The curvature of $\gamma(w)$ is computed as follows.

- $\dot{\gamma} = vt$
- $\dot{t} = \frac{\partial t}{\partial s}\frac{\partial s}{\partial w} = t'v$
- $\ddot{\gamma} = \frac{\partial}{\partial w}(vt) = \dot{v}t + v\dot{t} = \dot{v}t + v(t'v) = \dot{v}t + v^2t' = \dot{v}t + \kappa v^2 n$
- $\dot{\gamma} \times \ddot{\gamma} = (vt) \times (\dot{v}t + \kappa v^2 n) = \kappa v^3 t \times n$
- $||\dot{\gamma} \times \ddot{\gamma}|| = \kappa v^3 ||t \times n|| = \kappa v^3$
- $\kappa = \frac{||\dot{\gamma} \times \ddot{\gamma}||}{v^3} = \frac{||\dot{\gamma} \times \ddot{\gamma}||}{||\dot{\gamma}||^3}$

Torsion

- The binormal vector is $b = t \times n$.
- b' is orthogonal to t and to b.
  - $b \cdot t = 0$ implies $(b \cdot t)' = b' \cdot t + b \cdot t' = 0$, so $b' \cdot t = -b \cdot \kappa n = 0$.
  - $b \cdot b = 1$ implies $(b \cdot b)' = 2b' \cdot b = 0$.
- Define $b' = -\tau n$ with τ called the torsion.
- τ measures the deviation of the curve from the tn plane.
- A curve is planar if and only if τ is identically zero.
- $n' = (b \times t)' = b' \times t + b \times t' = (-\tau n) \times t + b \times (\kappa n) = \tau b - \kappa t$.
- The torsion formula is similar to the curvature formula, but contains the third derivative of γ.

Frenet Frame



- Frame: tangent t, principal normal n, and binormal b.
- They satisfy the ordinary differential equations

$$\begin{aligned} t' &= \kappa n \\ n' &= -\kappa t + \tau b \\ b' &= -\tau n \end{aligned}$$

- For given functions $\kappa(s)$ and $\tau(s)$, the Frenet equations determine the curve up to a translation and a rotation.

Generalized Frenet Equations

The Frenet equations for a general curve $\gamma(w)$ have an extra factor of $v = ||\dot{\gamma}||$.

$$\begin{aligned} \dot{t} &= v\kappa n \\ \dot{n} &= -v\kappa t + v\tau b \\ \dot{b} &= -v\tau n \end{aligned}$$

The first equation and the chain rule yield a useful formula for $\ddot{\gamma}$.

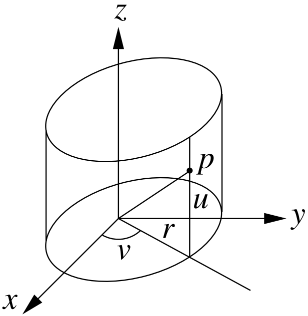
$$\ddot{\gamma} = \frac{\partial}{\partial w}(\dot{\gamma}) = \frac{\partial}{\partial w}(vt) = \dot{v}t + v\dot{t} = \dot{v}t + \kappa v^2 n$$

The tangential acceleration $\dot{v}t$ is due to the change in speed. The normal acceleration $\kappa v^2 n$ is due to the change in tangent direction.

Surfaces

- Representation
  - explicit: $z = f(x, y)$
  - implicit: $f(x, y, z) = 0$
  - parametric: $f(u, v) = (x(u, v), y(u, v), z(u, v))$
- Explicit surfaces are a special case of parametric surfaces.
- Implicit surfaces are more general, but often less convenient.
- An implicit or parameteric surface has an explicit representation in the neighborhood of a regular point.

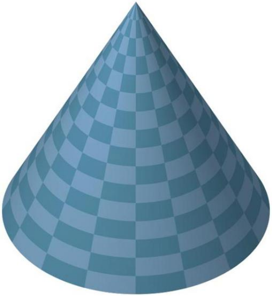
Cylinder

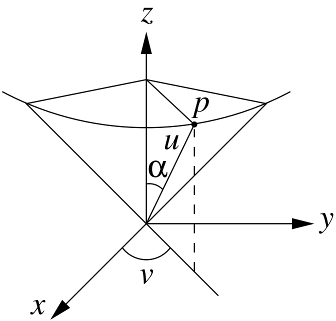


$$
\begin{aligned}
x(u,v) &= r\cos v \\
y(u,v) &= r\sin v \\
z(u,v) &= u
\end{aligned}
$$

Implicit representation: $x^2 + y^2 = r^2$.

Cone

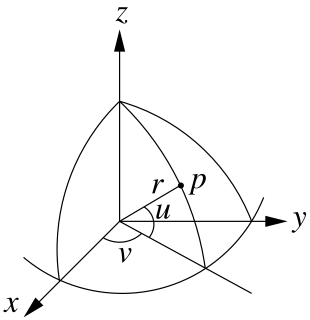




$$
\begin{aligned}
x(u,v) &= u \sin\gamma \cos v \\
y(u,v) &= u \sin\gamma \sin v \\
z(u,v) &= u \cos\gamma
\end{aligned}
$$

Implicit representation: $x^2 + y^2 = (z \tan\gamma)^2$.

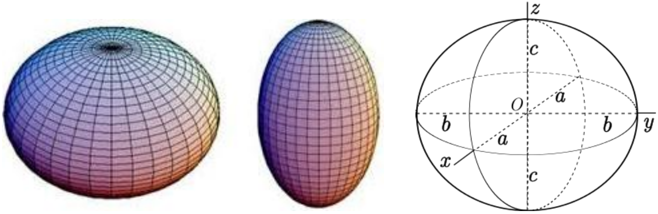
Sphere



$$\begin{aligned} x(u,v) &= r \cos u \cos v \\ y(u,v) &= r \cos u \sin v \\ z(u,v) &= r \sin u \end{aligned}$$

Implicit representation: $x^2 + y^2 + z^2 = r^2$.

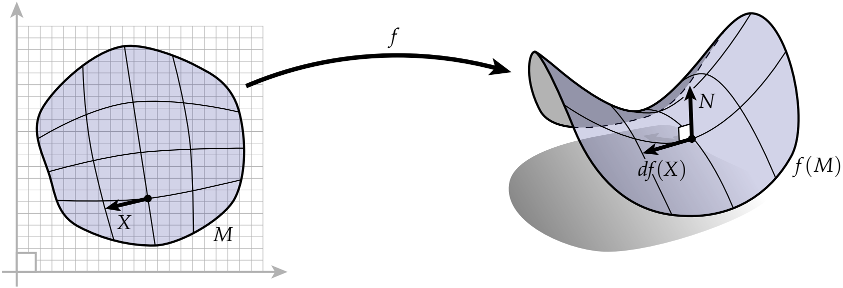
Ellipsoid



$$
\begin{aligned}
x(u,v) &= a\cos u\cos v\\
y(u,v) &= b\cos u\sin v\\
z(u,v) &= c\sin u
\end{aligned}
$$

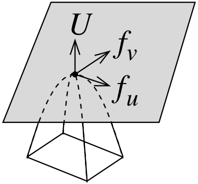
Implicit representation: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1.$

Parametric Surface



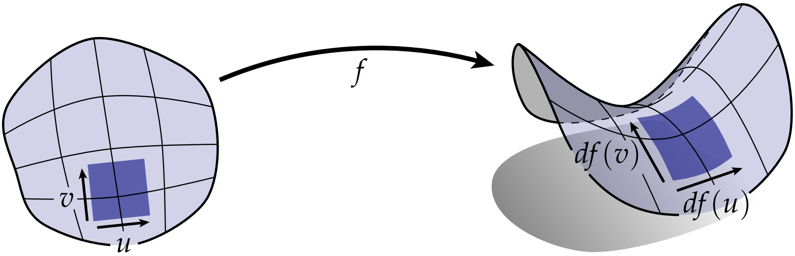
- The function f maps the parameter space M to the surface.
- The differential df maps a tangent vector X at $(u, v) \in M$ to a tangent vector $df(X)$ of the surface at $f(u, v) \in f(M)$.
- The surface is regular at points where df has full rank.
- We assume regularity from here on.

Tangent Plane



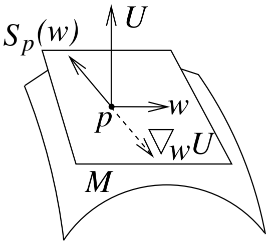
- The coordinate form of the differential is $df(u, v) = f_u u + f_v v$ with $f_u = \dfrac{\partial f}{\partial u}$ and $f_v = \dfrac{\partial f}{\partial v}$.
- The tangent vectors form a plane that is spanned by f_u and f_v.
- The tangent plane equation is $f(u, v) + f_u \Delta u + f_v \Delta v$.
- The unit normal is $U = f_u \times f_v / ||f_u \times f_v||$.
- A parameter space curve $(u(w), v(w))$ defines a spatial curve $g(w) = f(u(w), v(w))$ on $f(u, v)$ with tangent $\dot{g} = f_u \dot{u} + f_v \dot{v}$.

Angles and Area



- The angle between two curves with tangents $\dot{g}_1$ and $\dot{g}_2$ is given by $\cos\theta = \frac{\dot{g}_1}{||\dot{g}_1||} \cdot \frac{\dot{g}_2}{||\dot{g}_2||}$.
- The iso-parametric curves at (u, v) are $(u + w, v)$ and $(u, v + w)$ with tangent vectors f_u and f_v.
- The iso-parametric box with corners (u, v) and $(u + \Delta u, v + \Delta v)$ bounds an area of about $||f_u \times f_v||\Delta u \Delta v$.
- The differential of area is $||f_u \times f_v||$.

Shape Operator



- The shape operator of a surface M at a point p measures the deviation from the tangent plane in a direction w.
- The deviation equals the covariant derivative of the unit normal U in the w direction, denoted $\nabla_w U$.
- The shape operator is $S_p(w) = -\nabla_w U$.
- The minus sign simplifies some formulas.
- $S_p(w)$ is written as $S(w)$ in contexts where p is obvious.

Shape Operator Properties

Claim S_p is a symmetric linear operator on the tangent space at p.

1. Linearity is a standard property of the covariant derivative.

2. To show that $S(w)$ is orthogonal to U, differentiate $U \cdot U = 1$ to obtain $0 = w[U \cdot U] = 2U \cdot \nabla_w U = -2U \cdot S(w)$.

3. Symmetry means that $S(a) \cdot b = S(b) \cdot a$ for all tangent vectors a and b. It suffices to prove $S(f_u) \cdot f_v = S(f_v) \cdot f_u$ by linearity.

We have $S(f_u) = -\nabla_{f_u} U = -U_u$ and $S(f_v) = -\nabla_{f_v} U = -U_v$.

Differentiating $U \cdot f_u = 0$ yields
$\frac{\partial}{\partial v}(U \cdot f_u) = U_v \cdot f_u + U \cdot f_{uv} = -S(f_v) \cdot f_u + U \cdot f_{uv} = 0$
and so $S(f_v) \cdot f_u = U \cdot f_{uv}$.

Differentiating $U \cdot f_v = 0$ yields
$\frac{\partial}{\partial u}(U \cdot f_v) = U_u \cdot f_v + U \cdot f_{uv} = -S(f_u) \cdot f_v + U \cdot f_{uv} = 0$
and so $S(f_u) \cdot f_v = U \cdot f_{uv}$.

Shape Operator Matrix

We compute the matrix $A = [a_{ij}]$ of S in the $\{f_u, f_v\}$ basis. The columns of A are $S(f_u)$ and $S(f_v)$

$$\begin{aligned} a_{11} f_u + a_{21} f_v &= S(f_u) = -U_u \\ a_{12} f_u + a_{22} f_v &= S(f_v) = -U_v \end{aligned}$$

Taking dot products with f_u and f_v yields four equations.

$$\begin{aligned} a_{11} f_u \cdot f_u + a_{21} f_v \cdot f_u &= -U_u \cdot f_u \\ a_{11} f_u \cdot f_v + a_{21} f_v \cdot f_v &= -U_u \cdot f_v \\ a_{12} f_u \cdot f_u + a_{22} f_v \cdot f_u &= -U_v \cdot f_u \\ a_{22} f_u \cdot f_v + a_{22} f_v \cdot f_v &= -U_v \cdot f_v \end{aligned}$$

The matrix form of these equations is

$$\begin{bmatrix} f_u \cdot f_u & f_u \cdot f_v \\ f_u \cdot f_v & f_v \cdot f_v \end{bmatrix} \times \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -U_u \cdot f_u & -U_v \cdot f_u \\ -U_u \cdot f_v & -U_v \cdot f_v \end{bmatrix}$$

Shape Operator (continued)

Define $E = f_u \cdot f_u$, $F = f_v \cdot f_v$, and $G = f_v \cdot f_v$. Diffentiate $U \cdot f_v = 0$ and $U \cdot f_u = 0$ with respect to u and v to obtain.

$$\begin{aligned} L &= U \cdot f_{uu} = -U_u \cdot f_u \\ M &= U \cdot f_{uv} = -U_v \cdot f_u = -U_u \cdot f_v \\ N &= U \cdot f_{vv} = -U_v \cdot f_v \end{aligned}$$

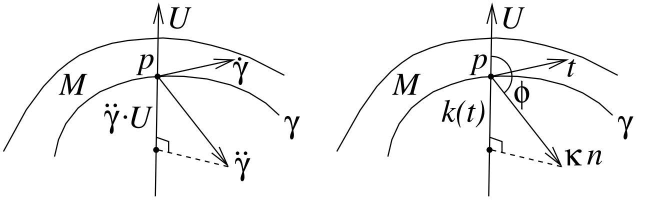
Substitute into the above matrix equation.

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} \times \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$$

Solve

$$A = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \times \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \frac{1}{EG - F^2} \begin{bmatrix} MF - LG & NF - MG \\ LF - ME & MF - NE \end{bmatrix}$$

Normal Curvature



(a) (b)

- (a) A curve γ in M satisfies $\ddot\gamma \cdot U = \dot\gamma \cdot S(\dot\gamma)$.
  Proof: $\dot\gamma \cdot U = 0$ implies $\ddot\gamma \cdot U = -\dot\gamma \cdot \dot U = \dot\gamma \cdot S(\dot\gamma)$.
- Every curve with velocity $\dot\gamma$ has the same normal acceleration.
- The curves with unit velocity t provide a canonical formula.
- The normal curvature is defined as $k(t) = S(t) \cdot t$.
- (b) Let γ have velocity t, curvature κ, and normal n.
- $k(t)$ is the projection of κn onto U.
- $k(t) = \kappa n \cdot U = \kappa \cos\phi$ with ϕ the angle between U and n.

Normal Curvature Computation

The shape operator in the direction $x = f_u u + f_v v$ is

$$S(f_u u + f_v v) = uS(f_u) + vS(f_v) = -U_u u - U_v v$$

Using L, M, and N from above,

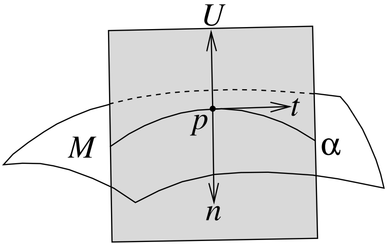
$$\begin{aligned} S(x) \cdot x &= -(U_u u + U_v v) \cdot (f_u u + f_v v) \\ &= -U_u \cdot f_u u^2 - (U_u \cdot f_v + U_v \cdot f_u) uv - U_v \cdot f_v v^2 \\ &= Lu^2 + 2Muv + Nv^2 \end{aligned}$$

The normal curvature in the direction x is

$$S\left(\frac{x}{||x||}\right) \cdot \left(\frac{x}{||x||}\right) = \frac{S(x) \cdot x}{x \cdot x} = \frac{Lu^2 + 2Muv + Nv^2}{Eu^2 + 2Fuv + Gv^2}$$

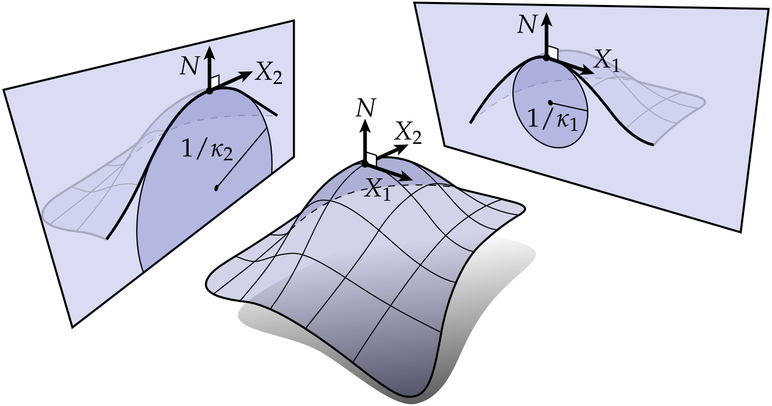
using E, F, and G from above.

Normal Section



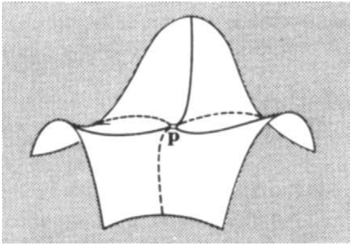
- The normal section in direction t is the intersection curve γ of M with the plane through p and tangent to t and to U.
- The curve normal n is collinear with the surface normal U.
- The normal curvature is $k(t) = \pm\kappa$.
- The curve γ provides a good visualization of $k(t)$. It curves away from U when $k(t) < 0$ and toward U when $k(t) > 0$.

Principal Directions and Curvatures



- The shape operator has real eigen values k_1 and k_2 with eigen vectors X_1 and X_2 because it is symmetric.
- If $k_1 > k_2$, the normal curvature has a maximum of k_1 in direction X_1 and a minimum of k_2 in direction X_2.
- These are called the principal directions and curvatures.

Umbilicals



monkey saddle

- A point with $k_1 = k_2$ is called an umbilical.
- The normal curvature is equal in all directions.
- Every point on a plane is an umbilical with $k = 0$.
- Every point on an r-sphere is an umbilical with $k = 1/r$.
- The monkey saddle has an isolated umbilical p with $k = 0$ where three zero-curvature curves meet.

Gaussian and Mean Curvature

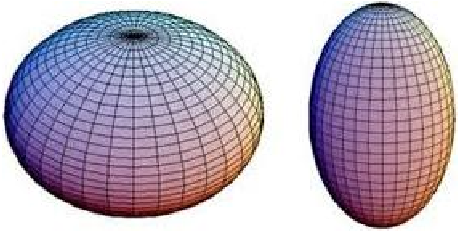
- The Gaussian curvature is $K = k_1 k_2$.
- It is the determinant of the shape operator

$$|A| = \left| \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \right| \times \left| \begin{bmatrix} L & M \\ M & N \end{bmatrix} \right| = \frac{LN - M^2}{EF - G^2}$$

- The sign of K determines the local shape.
- Surfaces with $K = 0$ are generated by sweeping a line.
- The mean curvature is $H = (k_1 + k_2)/2$.
- Surfaces with $H = 0$ minimize surface area.

Elliptic Point

sphere

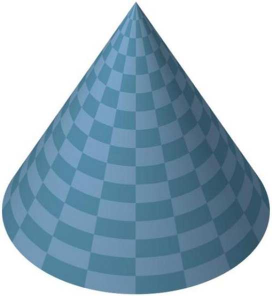


ellipsoid

- The Gaussian curvature is positive.
- The principal curvatures have the same sign.
- The surface lies on one side of the tangent plane.

Parabolic Point

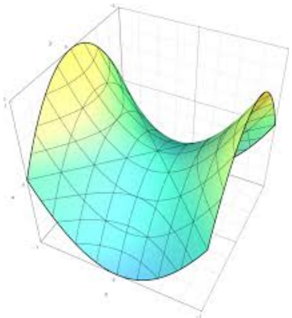
cylinder



cone

- The Gaussian curvature is zero.
- One principal curvature is zero.
- The surface intersects the tangent plane in a line.

Hyperbolic Point

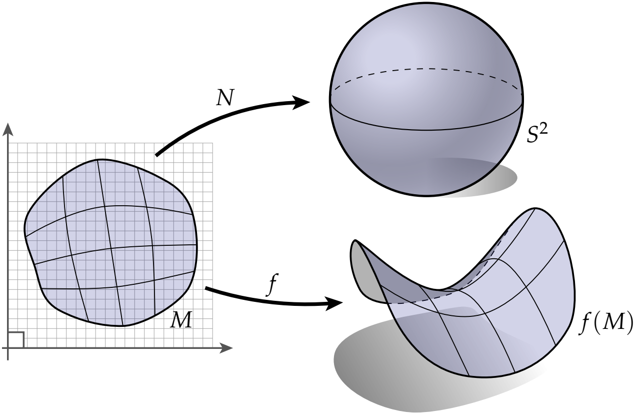


hyperbolic paraboloid

torus

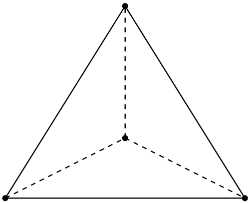
- The Gaussian curvature is negative.
- The principal curvatures have opposite signs.
- The surface intersects the tangent plane in two curves.

Gauss Map

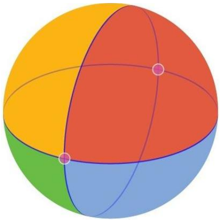


- The Gauss map $N : M \to S^2$ maps a point (u, v) in the parameter space M of a surface to the unit normal at $f(u, v)$.
- The area of the image of N equals the integral of the Gaussian curvature over M.
- This quantity is called the total Gaussian curvature.

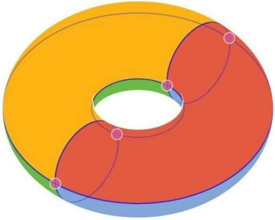
Euler Characteristic



$v = 4, e = 6, f = 4$



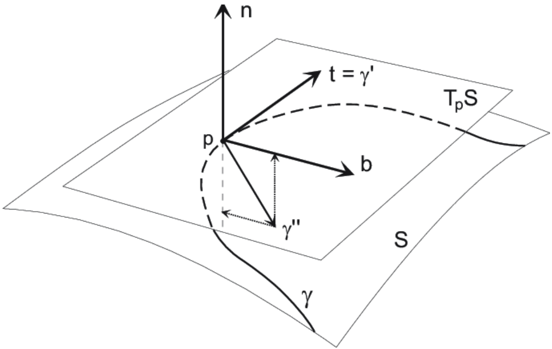
$v = 2, e = 4, f = 4$



$v = 4, e = 8, f = 4$

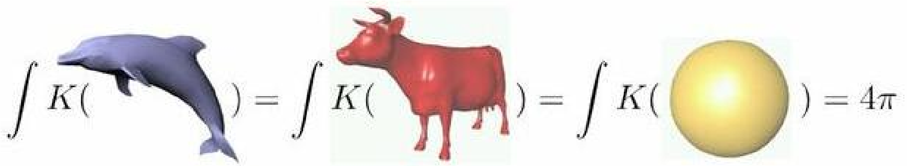
- The Euler characteristic of a polyhedron A with v vertices, e edges, and f facets is $\chi(A) = v - e + f$.
- Likewise for a smooth or piecewise smooth surface.
- $\chi = 2$ for a sphere and $\chi = 0$ for a torus.
- The Euler characteristic is a topological invariant.
- A compact, oriented, boundaryless surface is homeomorphic to a sphere with $k \geq 0$ handles and has $\chi = 2 - 2k$.

Tangential Curvature



- A curve γ on a surface S has Frenet frame t, n, and b.
- We have studied the normal curvature $k_n = n \cdot \gamma''$.
- We will now study the geodesic curvature $k_g = b \cdot \gamma''$.
- k_g is the complement of k_n because $t \cdot \gamma'' = t \cdot t' = 0$.
- k_g measures the acceleration tangent to S.

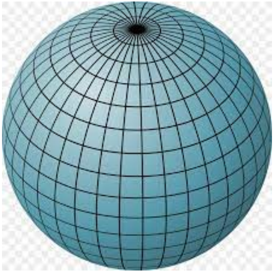
Gauss-Bonnet Theorem



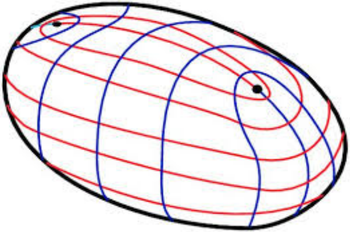
A surface M with boundary δM satisfies

$$\int_M K dA + \int_{\delta M} k_g ds = 2\pi\chi(M)$$

Geodesics



sphere



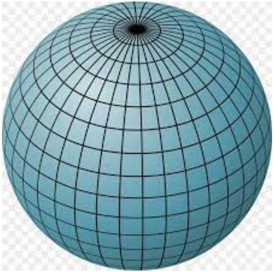
ellipsoid

A curve γ on $[a, b]$ is a *geodesic* if $k_g(p) = 0$ for all $p \in [a, b]$. It is a straightest curve in S.

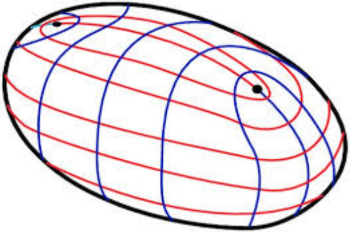
Equivalently, γ is a critical point of the length functional L with respect to tangential variations. For ϕ a tangent vector field along γ with $\phi(a) = 0$ and $\phi(b) = 0$, $\frac{\partial}{\partial \epsilon} L(\gamma + \epsilon\phi) = 0$.

γ is a locally shortest curve: every point on γ has a neighborhood in which γ is the shortest curve between every pair of its points.

Geodesic Completeness



sphere



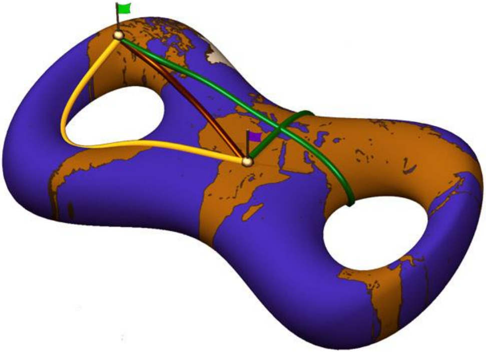
ellipsoid

There is a unique geodesic through every point p of a surface in every tangent direction t.

The geodesic is the solution of the ODE $\gamma'' = 0$ with initial conditions $\gamma(0) = p$ and $\gamma'(0) = t$.

A surface is complete if every geodesic can be extended indefinitely: it is periodic or converges to a boundary point.

Hopf-Rinow Theorem



The intrinsic distance between two points on a surface is the infimum of the lengths of the surface curves that connect them.

Hopf-Rinow Theorem Every pair of points on a geodesically complete surface is connected by a geodesic whose length equals the intrinsic distance between them.

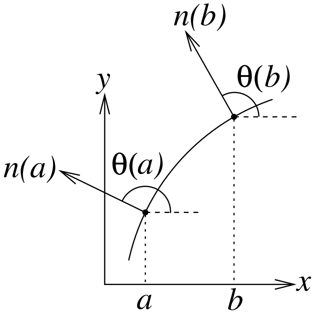
Discrete Differential Geometry

- Discrete differential geometry generalizes differential geometry to topological manifolds.
- The primary case is polyhedral surfaces.
- We will study their Gaussian curvature and geodesics.

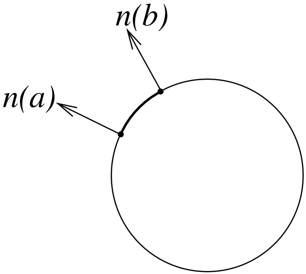
Discrete Differential Geometry: An Applied Introduction, Keenan Crane, online.

Straightest Geodesics on Polyhedral Surfaces, Polthier and Schmies, Siggraph 2006.

Curvature in the Plane



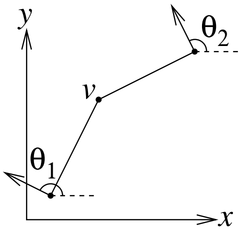
curve



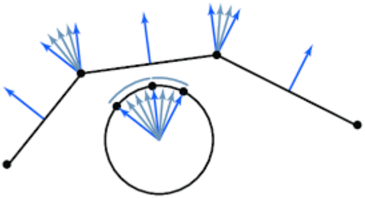
Gauss map

- A plane curve $\gamma : [a, b] \to \Re^2$ parameterized by arc length with normal angle θ and curvature k satisfies $\theta' = k$.
- The curvature is the rate of change of the normal angle.
- The total curvature of γ is $\int_a^b k = \theta(b) - \theta(a)$.
- This equals the length of the image of the Gauss map of γ.

Discrete Curvature in the Plane



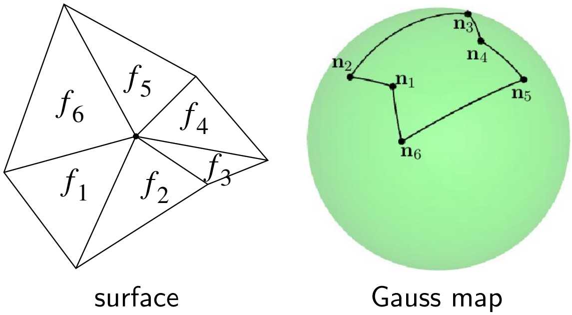
vertex curvature



total curvature

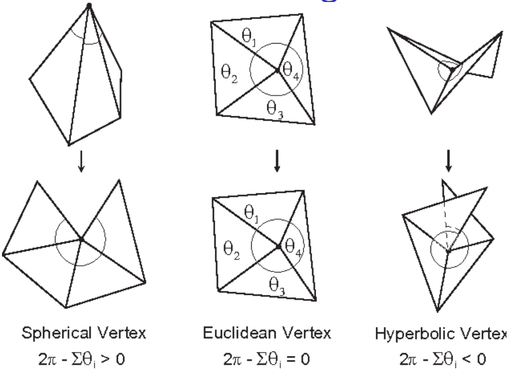
- Discrete differential geometry defines curvature on poly-lines.
- The curvature of a vertex is its change in normal angle: the outgoing angle minus the incoming angle, e.g. $k(v) = \theta_2 - \theta_1$.
- The curvature is zero elsewhere.
- The total curvature of a poly-line equals the length of the image of its Gauss map, as in the case of a smooth curve.

Gauss Map of a 3D Triangle Mesh



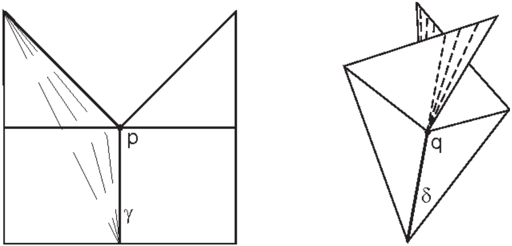
- Discrete differential geometry defines Gauss maps and Gaussian curvature for 3D triangle meshes.
- A face maps to its normal as before.
- an edge maps to the great circle arc bounded by the normals of the incident faces.
- A vertex maps to the spherical polygon bounded by the arcs of the incident edges.

Gaussian Curvature of a 3D Triangle Mesh



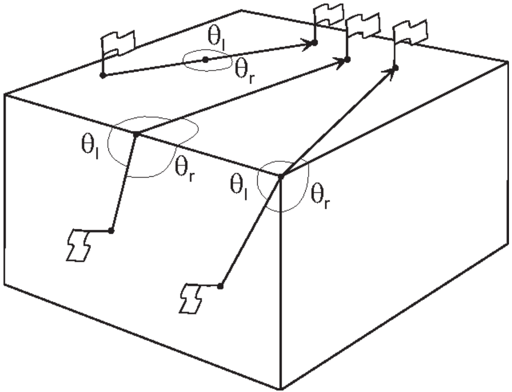
- The Gaussian curvature of a point in a 3D triangle mesh is defined as the area of its image in the Gauss map of the mesh.
- The Gaussian curvature is zero on edges and faces.
- The Gaussian curvature of a vertex whose incident faces have interior angles $\theta_i, \ldots, \theta_n$ is $2\pi - \sum_{i=1}^{n} \theta_i$.
- The total Gaussian curvature equals the area of the image of the Gauss map by construction.

Geodesics on Triangle Meshes



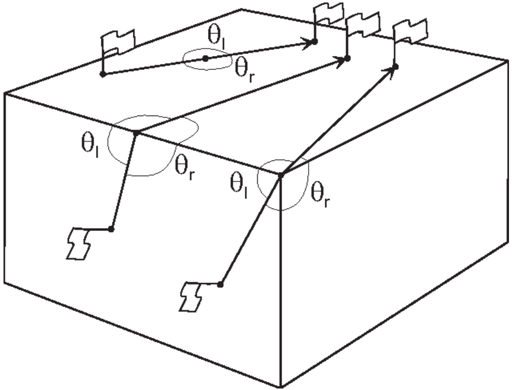
- Geodesics on triangle meshes are piecewise linear with breakpoints at vertices and on edges.
- The geodesics can be shortest curves or straightest curves.
- There is a fast algorithm for computing a shortest geodesic between two points.
- There is no shortest geodesic through a spherical vertex.
- There are a continuum through a hyperbolic vertex.

Straightest Geodesics



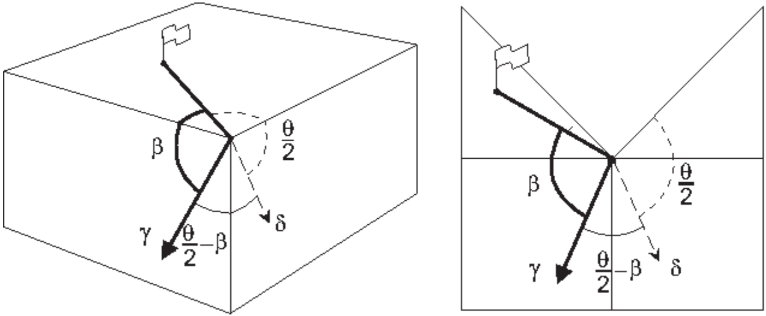
- Let p be a point on a piecewise linear curve in a triangle mesh.
- The angle of p is $\theta = 2\pi - c$ with c the curvature of p.
- The curve has left and right angles θ_l and θ_r with $\theta_l + \theta_r = \theta$.
- The curve is a straightest geodesic if $\theta_l = \theta_r$ at every p.

Straightest Geodesics (continued)



- There is a unique straightest geodesic through every point in every direction.
- There can be no straightest geodesic between two points.
- Shortest and straightest geodesics differ solely at vertices.

Straightest Geodesic Curvature

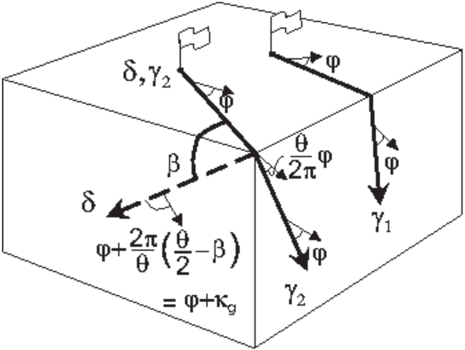


- A point p with angle θ lies on a curve γ.
- The straightest geodesic at p with direction γ' is δ.
- The straightest geodesic curvature of γ is the normalized angle between γ and δ: $k_g = \frac{2\pi}{\theta}(\frac{\theta}{2} - \beta)$ with $\beta = \theta_l$.
- Setting $\beta = \theta_r$ reverses the sign of k_g.
- A curve is a straightest geodesic iff $k_g = 0$ at every point.

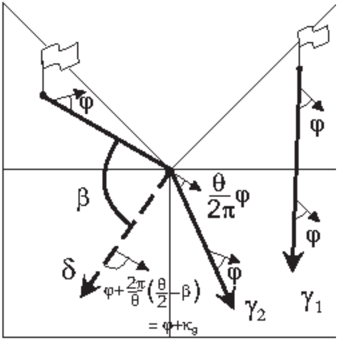
Parallel Translation

- Integration rules combine tangent vectors at multiple points.
- This operation is trivial in Euclidean spaces.
- Tangent vectors on a smooth surface can be combined in the ambient Euclidean space.
- The integration rules must be modified to stay on the surface.
- Vectors on polyhedral surfaces are transferred to a common base point via parallel translation.
- The resulting integration rules stay on the surface.

Parallel Translation



geodesic curve



general curve

A tangential vector field v is parallel along a straightest geodesic γ if the normalized angle between $v(s)$ and $\gamma'(s)$ is constant.

A tangent vector v_0 with normalized angle α_0 on a general curve γ defines a unique parallel vector field $\alpha(s) = \alpha_0 + \int_0^s k_g$.