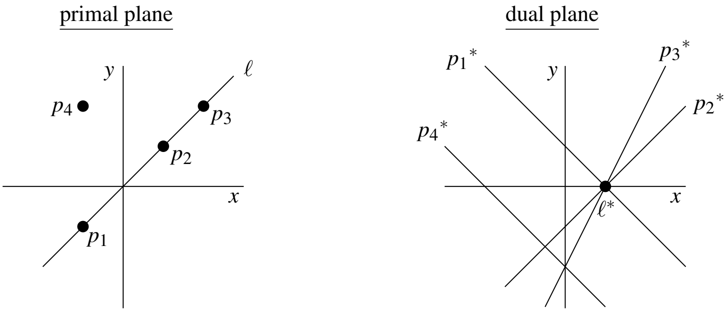
Duality (Sections 8.2, 11.4, and 11.5; Cheng 2.3)

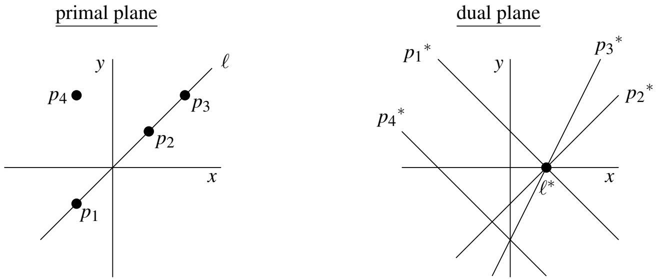
Elisha Sacks

Duality (Sec. 8.2)



- The dual of a point $p = (p_x, p_y)$ is the line $p^* = p_x x - p_y$.
- The dual of a line $\ell = l_a x + l_b$ is the point $\ell^* = (l_a, -l_b)$.
- A vertical line does not have a dual.
- The dual of the dual is the original: $(p^*)^* = p$ and $(\ell^*)^* = \ell$.

Properties of Duality

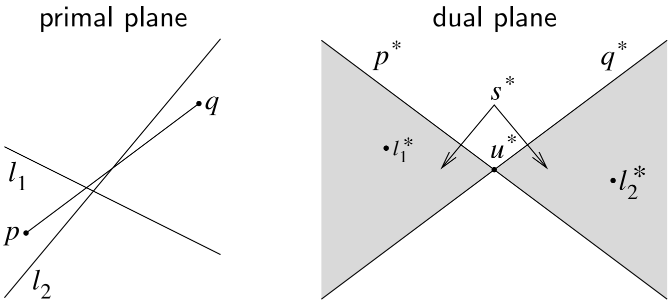


Property A point p is on a line ℓ iff ℓ^* is on p^*.
Proof The primal equation is $l_a p_x + l_b = p_y$ and the dual equation is $p_x l_a - p_y = -l_b$.

Corollary Points $p_1, \ldots, p_n$ lie on a line ℓ iff ℓ^* is the common intersection point of the lines $p_1^*, \ldots, p_n^*$.

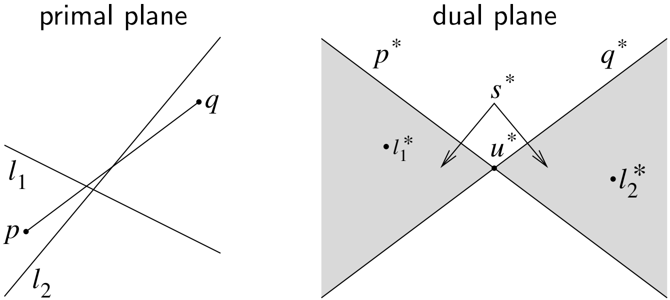
Property A point p is above a line ℓ iff ℓ^* is above p^*.
Proof The primal equation is $l_a p_x + l_b < p_y$ and the dual equation is $p_x l_a - p_y < -l_b$.

Line Segment Duality



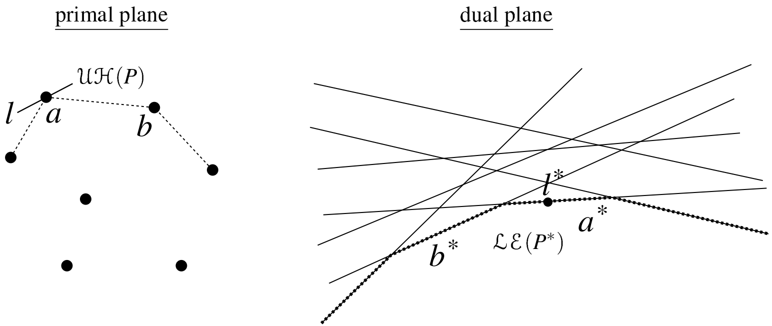
- The dual of a segment $s = pq$ with $p_x < q_x$ is $s^* = \cup_{a \in s} a^*$.
- Let the pq line u be $y = ax + b$.
- We have $p = (p_x, ap_x + b)$ and $q = (q_x, aq_x + b)$.
- The lines p^* and q^* intersect at $u^* = (a, -b)$.
- s^* is the wedge between p^* and q^*.

Line Segment Duality



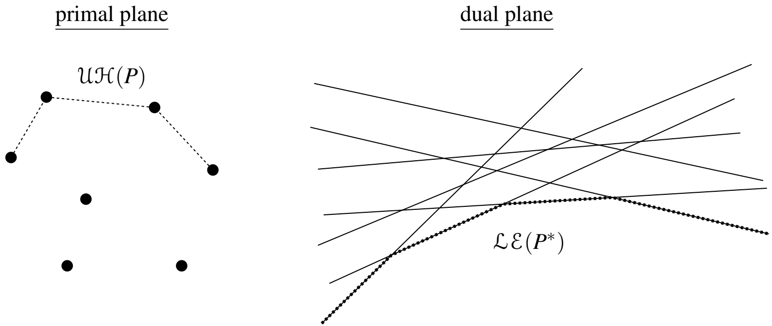
- A line l intersects s iff s^* contains l^*.
- Lines above p and below q map to the left half of the wedge.
- Lines below p and above q map to the right half of the wedge.
- When $p_x \to -\infty$ and $q_x \to \infty$, p^* and q^* become vertical and the wedge converges to the entire plane.

Duality of Upper Hull and Lower Envelope (Sec. 11.4)



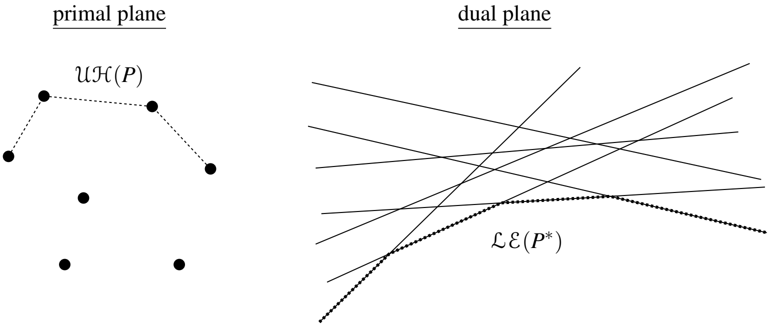
- A point $a \in P$ is in the upper hull of P iff there is a line l through a that is above every other point $p \in P$.
- The point l^* is on the line a^* and below every other $p^* \in P^*$.
- The line a^* contains an edge of the lower envelope.
- As l rotates clockwise, l^* traverses the edge from right to left.
- When l is the supporting line of a hull edge ab, l^* is the envelope vertex $a^* \cap b^*$.

Duality of Upper Hull and Lower Envelope (continued)



- The upper hull vertices are in increasing x order.
- The corresponding lower envelope edges are in increasing slope order from right to left.
- The first and last vertices correspond to unbounded edges.

Duality of Upper Hull and Lower Envelope (concluded)



- The lower hull corresponds to the upper envelope of P^*.
- The two hulls have the same left and right points p_l and p_r.
- The two envelopes are disjoint.
- p_l^* and p_r^* contain unbounded edges in both envelopes.
- Full duality occurs in the projective plane.

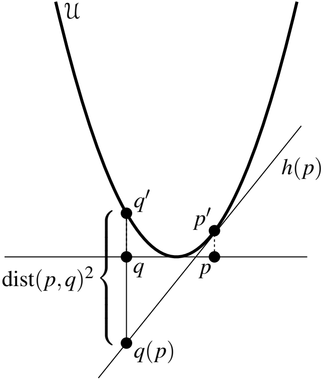
Duality in 3D

- The dual of $p = (p_x, p_y, p_z)$ is the plane $z = p_x x + p_y y - p_z$.
- A plane parallel to the z axis has no dual.
- The dual of the supporting line of pq is the line $p^* \cap q^*$.
- A point p is on/above a plane l iff l^* is on/above p^*.
- Points $p_1, \ldots, p_n$ lie on a plane l iff l^* is the common intersection point of the planes $p_1^*, \ldots, p_n^*$.
- The upper hull is dual to the lower envelope.
  - a is a hull vertex iff a^* contains an envelope facet.
  - ab is a hull edge iff $a^* \cap b^*$ contains an envelope edge.
  - a, b, and c are coplanar iff $a^* \cap b^* \cap c^*$ is an envelope vertex.
  - Equivalently, a, b, and c are on the boundary of a hull facet.
  - The boundary vertices and edges of the hull are dual to the unbounded facets and edges of the envelope.

Voronoi Diagram and Delaunay Triangulation

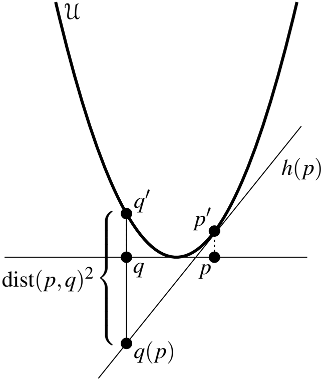
- The Voronoi diagram and the Delaunay triangulation in dimension d are derivable from the lower convex hull in dimension $d + 1$.
- We will study the derivation in dimension $d = 2$.
- The $d = 2$ algorithms are mainly of theoretical interest because simple optimal algorithms are already available.
- For $d > 2$, the convex hull derivations are the standard.

Voronoi Diagram (Sec. 11.5)



- The 2D Voronoi diagram is computed in the $z = 0$ plane.
- A point p lifts to the point $p' = (p_x, p_y, p \cdot p)$ on the paraboloid $z = x^2 + y^2$.
- The tangent plane $h(p)$ at p' is $z = 2p_x x + 2p_y y - p \cdot p$.

Voronoi Diagram (continued)



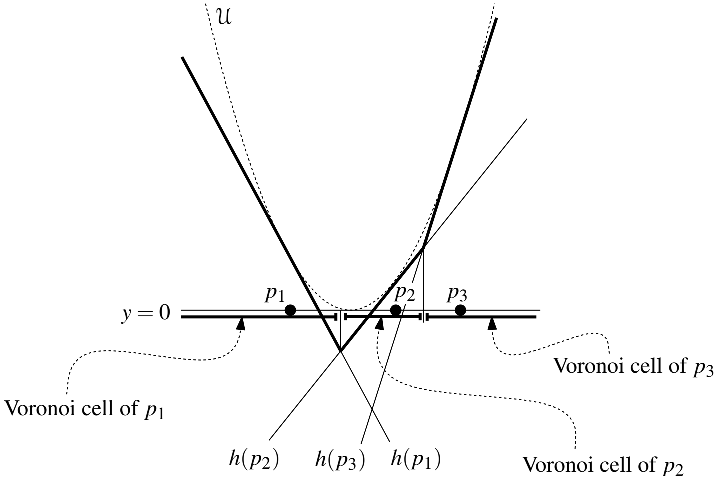
- Let q be a point in the Voronoi cell of p.
- The distance from q' to the point below, $q(p)$, on $h(p)$ is $q \cdot q - 2p \cdot q + p \cdot p = (p - q) \cdot (p - q) = ||p - q||^2$.
- The distance to any other site tangent plane is greater.
- q' is above the $h(p)$ facet of the upper envelope of the planes.

Voronoi Diagram Algorithm

Lift to $z = 0.5(x^2 + y^2)$, so p' is dual to $h(p)$.

1. Compute the lower hull of the lifted sites.
2. Construct the upper envelope of the tangent planes.
3. Project onto the $z = 0$ plane.

1D Voronoi Diagram from 2D Upper Envelope



Delaunay Triangulation from Lower Hull (Cheng 2.3)

Lemma 2.1 (Lifting Lemma) The lift of a circle c lies on a plane h. A point inside/outside c lifts to a point below/above h.

Proof Let o and r be the center and radius of c.

Expanding $||o-p||^2 = (o-p)\cdot(o-p)$ yields
$p'_z = ||p||^2 = 2o\cdot p - o\cdot o + ||o-p||^2$.

The equation $z(p) = 2o\cdot p - o\cdot o + r^2$ defines a plane h.

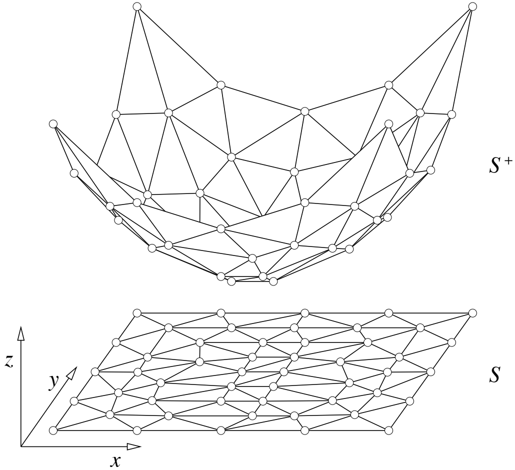
The vertical distance from p' to h is $||o-p||^2 - r^2$.

If p is on c, $||o-p|| = r$, so p' is on h.

If p is inside c, $||o-p|| < r$, so p' is below h.

If p is outside c, $||o-p|| > r$, so p' is above h.

Delaunay Triangulation Algorithm



- A triangle has an empty circle iff every other lifted point lies above the plane of the lifted triangle.
- The lifted triangle is a facet of the lower hull.
- The projection of the lower hull is the Delaunay triangulation.