Chapter 8: Nonlinear Equations

Nonlinear Equations

• Definition

A value for parameter x that satisfies the equation

$f(x) = 0$

is called a root or a ("zero") of $f(x)$.

• Exact Solutions

For some functions, we can calculate roots exactly;

e.g.,

- Polynomials up to degree 4
- Simple transcendental functions, such as

$$\sin x = 0$$

which has an infinite number of roots:

$$x = k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

• Numerical Methods

Used to estimate roots for nonlinear functions f(x).

- Bisection Method
- False Position Method
- Newton's Method
- Secant Method

All are iterative techniques applied to continuous functions.

We will consider only methods for finding real roots, but methods discussed can be generalized for complex roots.

Bisection Method

A binary search procedure applied to an x interval known to contain root of $f(x)$.

Example: Polynomial $f(x) = x^5 + x + 1$

- has exactly five roots, at least one real root.

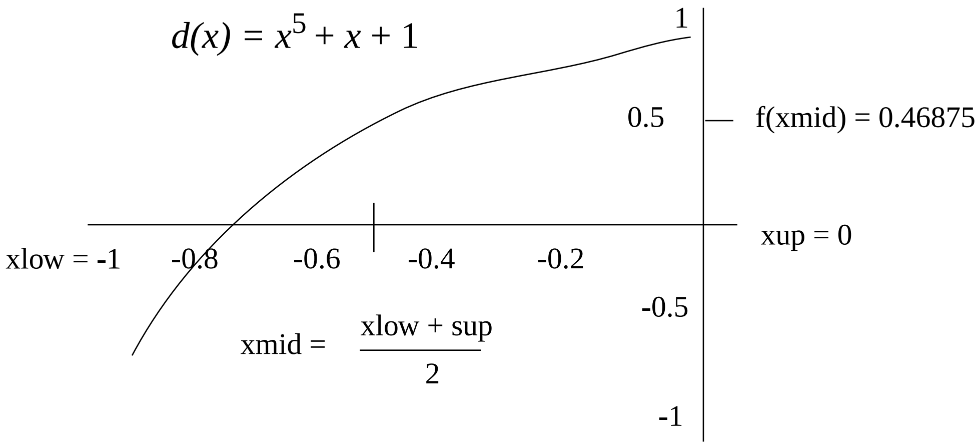
Step 1: Determine an x interval containing a real root.

We can do this by simple analysis of $f(x)$:

- compute (or estimate) f(x) for convenient values of x, such as $0, \pm 1$.
- estimate position of local extrema with first derivative of $f(x)$.
- make rough sketch of $f(x)$.

x	$f(x)$	
1	3.0	
0	1.0	Have root in this interval.
-1	-1.0	

Step 2: Bisect interval repeatedly until root is determined to desired accuracy.



- If f(xlow) and f(xmid) have opposite signs (f(xlow) . f(xmid) < 0,
  root is in left half of interval.

- If f(xlow) and f(xmid) have same signs (f(xlow) . f(xmid) > 0),
  root is in right half of interval.

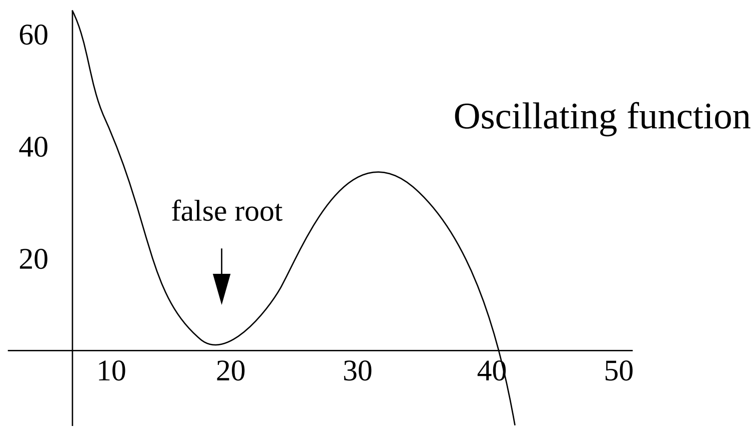
Continue subdividing until interval width has been reduced to a size $\leq \varepsilon$

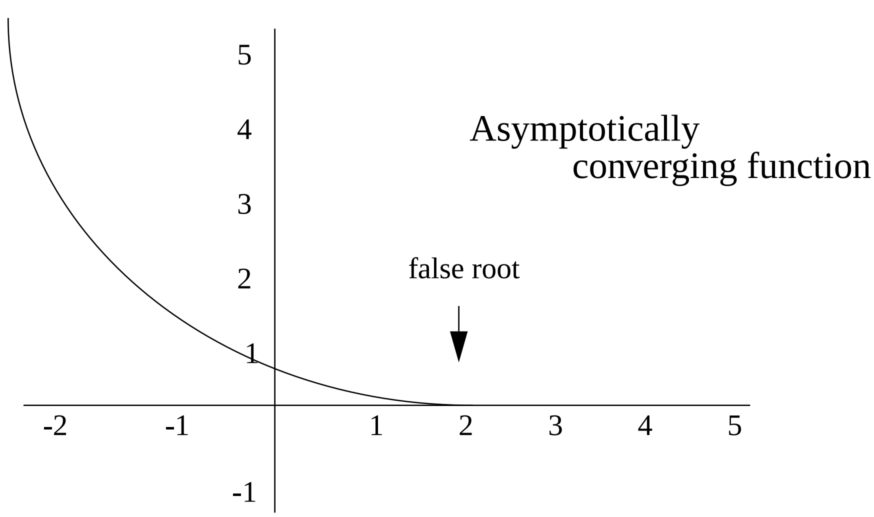
where

ε = selected x tolerance.

Using a *y* tolerance can result in poor estimate of root.

Examples:





<u>Pseudocode Algorithm: Bisection Method</u>

```
Input xLower, xUpper, xTol
yLower = f(xLower)                    (* invokes fcn definition *)
xMid = (xLower + xUpper)/2.0
yMid = f(xMid)
iters = 0                    (* count number of iterations *)
While ( (xUpper - xLower)/2.0 < xTol )
        iters = iters + 1
        if( yLower * yMid > 0.0) Then xLower = xMid
                Else xUpper = xMid
        Endofif
        xMid = (xLower + xUpper)/2.0
        yMid = f(xMid)
Endofwhile
Return xMid, yMid, iters (* xMid = approx to root *)
```

(Do not need to recalculate yLower in loop, since it can never change sign.)

For a given *x* tolerance (epsilon), we can calculate the number of iterations directly. The number of divisions of the original interval is the smallest value of *n* that satisfies:

$$\frac{x_{upper} - x_{lower}}{2^n} < \varepsilon \qquad or \qquad 2^n > \frac{x_{upper} - x_{lower}}{\varepsilon}$$

Thus $n > \log 2\left(\frac{x_{upper} - x_{lower}}{\varepsilon}\right)$

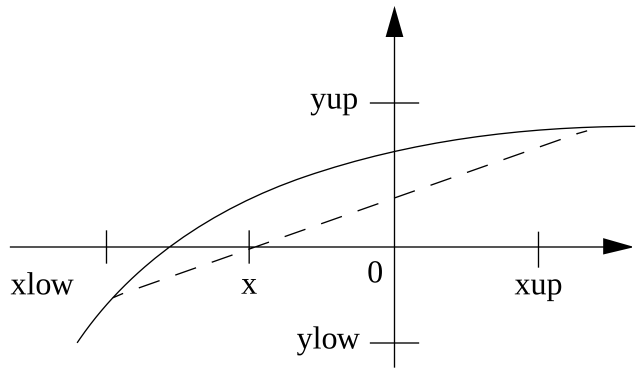
In our previous example $x_{lower} = -1$, $x_{upper} = 0$

Choosing $\varepsilon = 10^{-4}$, we have $n > \log_2 10^{-4} = 13.29$

$$\therefore n = 14$$

False-Position Method (Regula Falsi)

Improvement on bisection search by interpolating next *x* position, instead of having the interval.



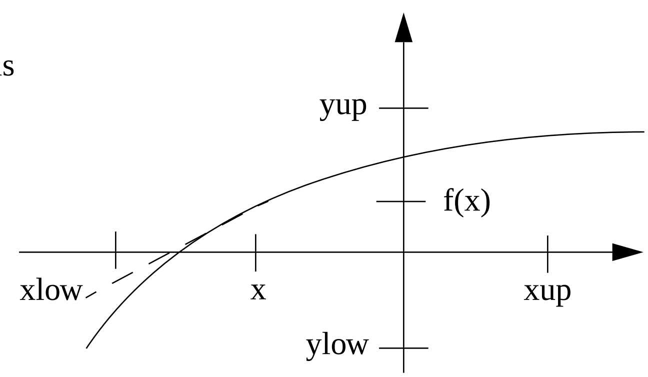
By similar triangles: $\frac{yup - 0}{xup - x} = \frac{0 - ylow}{x - xlow}$ or

$$\boxed{x = \frac{xlow \cdot yup - xup \cdot ylow}{yup - ylow}}$$

Then use this calculation in place of xmid in the Bisection Algorithm.

- False Position Method usually converges more rapidly than Bisection approach.
- Can improve false position method by adjusting interpolation line:

Each interpolation line (after first) is now drawn from (x, f(x)) to either (xlow, ylow/2) or to (xup, yup/2), depending on region containing root.

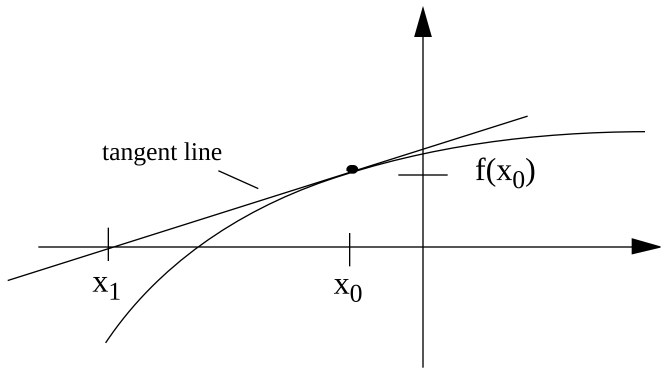


For continuous functions (single-valued), both Bisection and False Position are guaranteed to converge to root.

(Because interval is assumed to contain root.)

<u>Newton's Method</u> (Newton-Raphson)

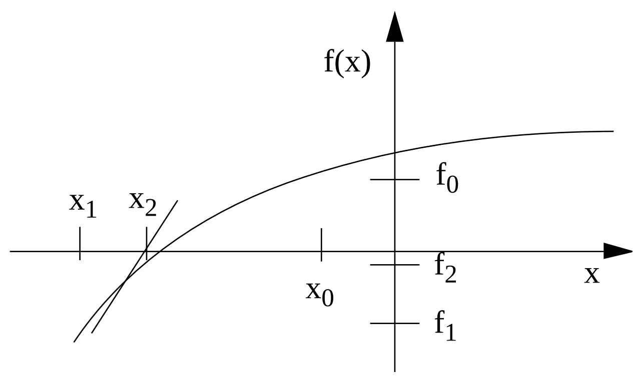
Attempts to locate root by repeatedly approximating *f(x)* with a linear function at each step:



Start with initial "guess", x_0, then calculate next approximation to root, x_1.

Slope of curve at x_0 is $\dfrac{df}{dx} = \dfrac{f(x_0)}{f'(x_0)}$

We repeat process to get next approximation, etc.



Thus, rapidly converges to root, with convergence accelerating as we approach *f(x)* = 0.

Newton-Raphson Algorithm

1. Start with an initial guess x_0 and an x-tolerance ε.
2. Calculate

$$x_k = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})}, \qquad k = 1, 2, 3, \ldots$$

   until

$$\left|\frac{f(x_{k-1})}{f'(x_{k-1})}\right| < \varepsilon$$

   Very fast root-finding method. Useful when $f'(x)$ is not too difficult to evaluate.

   But Newton's Method does not always converge; e.g., when $f'(x) \approx 0$ for some x.

   Example: Newton-Raphson Solution of

$$f(x) = x^2 - 2, \qquad x_0 = 1 \text{ (initial guess)}$$

derivative: $f'(x) = 2x$

Steps

(1)
$$f(x_0) = -1, \qquad f'(x_0) = 2$$
$$x_1 = 1 - (-1)/2 = 1.5000000$$

(2)
$$f(x_1) = 0.25, \qquad f'(x_1) = 3.00$$
$$x_2 = 1.5 - 0.25/3.00 = 1.4166667$$

(3)
$$f(x_2) \approx 0.0069444, \qquad f'(x_2) \approx 2.8333333$$
$$x_3 \approx 1.4166667 - 0.0069444/2.8333333 \approx 1.4142157$$

Continue until $|x_k - x_{k-1}| < e$.

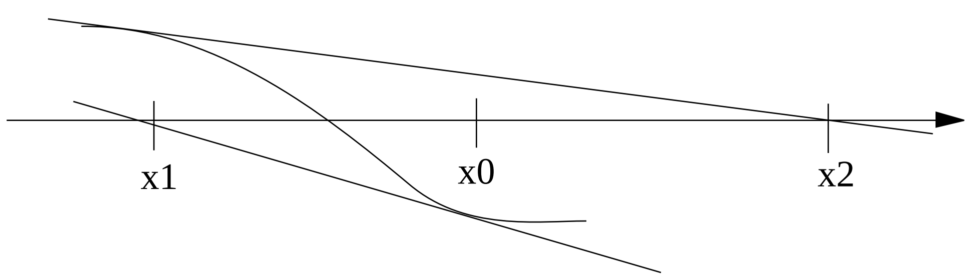
Pseudocode Algorithm - Newton's Method

```
Inputx0, xTol
iters = 1
dx = -f(0)/fDeriv(x0)    (* fcns f and fDeriv *)
root = x0 + dx
While (Abs(dx) ≥ xTol)
        dx = -f(root)/fDeriv(root)
        root = root + dx
        iters = iters + 1
Endofwhile
Return root, iters
```

Newton's Method

- Converges much faster than Bisection Method.

- But convergence not guaranteed. Will diverge in some cases. For example:



To prevent such "run aways", we can include the test: iters > maxiters in above algorithm to halt the loop.

- For multiple roots (real and complex), boundaries between convergence regions are fractals.

Secant Method

Variation of Newton's Method:

- Approximates derivative
- Requires two starting *x* values

Let x_k denote approximation to root of *f(x)* at step *k*.

Using Newton's Method, next approx. is

$$x_{k+1} = x_k - f(x_k) / f'(x_k)$$

Since the derivative

$$f'(x) = \lim_{Ax \to 0} \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

can be approximated as

$$f'(x_x) \approx \frac{f(x_x) - f(x_{k-1})}{x_k - x_{k-1}} = \frac{y_k - y_{k-1}}{x_k - x_{k-1}}$$

the expression for x_{k+1} can be approximated as

$$\boxed{x_{k+1} = \frac{x_{k-1} y_k - x_k y_{k-1}}{y_k - y_{k-1}}}$$

Thus, Secant Method based on using a "finite-difference" approximation for derivative.

<u>Example</u>: Secant Method Solution of

$$f(x) = x^2 - 2, \qquad x_0 = 1, x_1 = 2 \quad \text{(starting values)}$$

<u>Steps</u>

$$y_0 = f(x_0) = -1, \qquad y_1 = f(x_1) = 2$$

(1) $$x_2 = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0}$$

$$= \frac{(1)(2) - (2)(-1)}{3} = \frac{4}{3} \approx 1.3333$$

$$y_2 = f(x_2) \approx -0.2222$$

(2) $$x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1}$$

$$\approx \frac{(2)-(0.2222) - (1.3333)(2)}{(-0.2222) - (2)}$$

etc.

Pseudocode Algorithm - Secant Method

```
Input xk, xkMinus1, xTol, maxiters
iters = 1
yk = (xk)                    (* invokes function f *)
ykMinus1 = f(xkMinus1)
root = (xkMinus1*yk - xk*ykMinus1)/(yk - ykMinus1)
ykPlus1 = f(root)
While( (Abs(root - xk) ≥ xTol) and (iters ≤ maxiters) )
        xkMinus1 = xk
        ykMinus1 = yk
        xk = root
        yk = ykPlus1
        root = (xkMinus1*yk - xk*ykMinus1)/(yk - yk Minus1)
        ykPlus1 = f(root)
        iters = iters + 1
Endofwhile
Return root, ykPlus1, iters
```

Secant Method can be best choice if
computation of f'(x) is expensive

Summary of Root-Finding Methods

Method	Input	Converge?	Converge Rate
Bisection	xlow, xup	yes	linear
False Pos.	xlow, xup	yes	better
Secant	"any" 2 vals	maybe	better yet
Newton's	"any" 1 val	maybe	usually best

Root-Finding Methods in Mathematica

Numerical solution of equations can be accomplished with either `Solve` or `Roots` (for polynomials) in combination with **N** function.

Numerical Root-Finding using Newton or Secan t Method:

FindRoot[f(x)==expr, {x, x0}] - Newton's Method using starting value x0.

FindRoot[f(x)==expr, {x, x0, xmin, xmax}] - use starting value x0; stop if x goes outside range xmin to xmax.

FindRoot[{eqn1, eqn2, . . . }, {x, x0}, {y, y0}, . . .] - find numerical solution for a set of simultaneous equations; using starging values s0, x0, . . .

FindRoot[f(x)==expr, {x, {x0, x1}}] - Secant Method with starting values x0, x1.

Examples - Newton's Method

```
FindRoot[x^2-2 == 0, {x, -1}]
{x -> -1.41421}
```
{if starting value is neg., a neg. real root is sought. If pos., a pos. root is sought

```
FindRoot[x^2+2 == 0, {x, I}]
{x -> 1.41421 I}
```
{if starting value is complex, then Mathematica looks for a complex root.

```
FindRoot[x^3 == -1, {x, I}]
{x -> 0.5 + 0.866025 I}
```

```
FindRoot[{x^2-2Sin[y]==0,y Sqrt[x]-1==0},
     {x,1}, {x,1}]
```
{simultaneous eqns.

```
{x -> 1.24905, y -> 0.894767}
```

Parameters for FindRoot include <u>MaxIterations</u> and <u>AccuracyGoal</u>.
E.g.,

```
FindRoot[x^2-2==0, {x,1}, MaxIterations->3]

FindRoot::cvnwt:
  Newton’s method failed to converge to converge
  to the prescribed accuracy after 3 iterations.
{x-> 1.41422}
```
(Default MaxIterations=15)

(AccuracyGoal is number of digits required for 0.0 in solution check of f(x) = 0.0; default = 9(?); default Precision for calculations = 19.)

<u>Secant Method Examples:</u>

```
FindRoot[x^2-2 == 0, {x, {1, 2}} ]

{x -> 1.41421}
```
(starting values = 1,2)

```
FindRoot[x^2-2Sin[y]==0, y Sqrt[x]-1==0),
                {x,{1,2}}, {y, {1,2}}]

{x -> 1.24905, y -> 0.894767}
```

(starting values for both x and y are 1,2)

Note that FindRoot gives solutions as transformation rules (using ->). Thus the solutions can be substituted into expression with /. operatior. We can combine this with an assignment statement to store values in a variable name.

Example - Assign the positive square root of 9 to variable xroot:

```
xroot = x /.FindRoot[x^2 - 9 == 0, {x, 2}]

3.
```
(Initial value for Newton’s Method is 2)

Function `Solve` also gives solutions to an equation, or set of equations, as transformation rules, which may then be subtbituted into expressions using the /. operator.

Examples:

```
Solve[x^2-9==0, x]
{{x -> -3}, {x -> 3}}


NSolve[Cos[x]==0, x]
Solve::ifun:

Warning: Inverse functions are being used by
    Solve, so some solutions may not be found.

{{x -> 1.5708}}
```

Function ROots, on the other hand, gives roots of polynomials as logical expressions (i.e., with ==) that can be manipulated lwith Boolean operators.

Examples:

```
Roots[x^2-9==0, x]
x == 3 | | x == -3


NRoots[x^2-9==0, x]
x == -3. | | x == 3.
```

Summary

Mathematica Equation-Solving Functions

- `Roots` -

  Applied to polynomials.

  Solutions given as logical expressions, involving either variable names or numerical values.

  Can be used with **`N`** function.

- `Solve` -

  Applied to single equation or set of simultaneous equations.

  Solutions given as transformation rules, involving either variable names or numerical names.

  Can be used with **`N`** function.

- `FindRoot` -

  Applied to single equation or set of simultaneous equations.

  Uses either Newton's Method or Secant Method to find numerical solutions.

  Solutions given as transformation rules.