DECREASING THE DIAMETER OF CYCLES

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Abstract. Alon, Gyárfás and Ruszinkó [1] established a bound for the minimum number of edges that have to be added to a graph of $n \geq n_0$ vertices in order to obtain a graph of diameter 2. Alon et al. approximated n_0 by a polynomial function of the maximum degree of the initial graph. They conjectured that the minimum value for n_0 is 12 for the case of C_n, as opposed to $n_0 = 274$ obtained by calculations for the general case. In this paper we prove their conjecture.

For the reduction to diameter 3 of a cycle we also improve the bounds from [1] showing that the minimum number of edges that need to be added to the cycle C_n is between $n - 59$ and $n - 8$.

1. Introduction

In [1] Alon, Gyárfás, and Ruszinkó proved the following theorem.

Theorem 1. *Let G be a graph with n vertices of maximum degree D. Then at least $n - D - 1$ edges are needed to extend G into a graph of diameter at most 2, provided that $n \geq n_0$ for some n_0.*

A corollary of this theorem says that we need to add at least $n - 3$ edges to C_n in order to obtain a diameter 2 graph. Their proof approximates n_0 as $(D^2 + D + 1)(2D^3 + 5D^2 + 2D - 1) + 1$ which is 274 for a cycle. By checking for small n's it is apparent that $n_0 = 12$ for the C_n case.

Also Alon et al. proved that at least $n - 3(D+1)^3 - 2(D+1)^2 - 1$ edges are needed to extend a graph G to a graph of diameter three and conjectured that for the case of C_n there actually are at least $n - 6$ edges needed. By giving a construction of a graph of diameter 3 obtained from C_n by adding $n - 8$ edges, we prove that their conjecture is false.

2. Diameter 2

Theorem 2. *For $n \geq 12$ at least $n - 3$ edges must be added to C_n in order to obtain a graph of diameter 2.*

Proof. Our method of proof will be to consider various cases. Throughout this paper H will denote a graph that has $V(H) = V(C_n)$ such that $C_n \cup H$ has diameter 2; $d(x, y)$ will denote the distance between vertices x and y in $C_n \cup H$; $e(H)$ will denote the number of edges in H.

We leave for the reader to check that the theorem holds in the cases that follow. The idea is to count the number of edges that must exist among various pairs of vertices, such that $d(u, v) \leq 2$ for any $u, v \in V(C_n)$.

Case 1. H has a vertex a such that $deg_H(a) = 0$.

Case 2. There exists an edge in H both of whose end vertices have degree 1 in H.

Case 3. There exist two vertices adjacent in C_n of degree 1 in H.

We prove the remaining case by induction. Note that here we have no vertex of degree 0 in H, so if $e(H) \leq n - 4$ then there are at least eight vertices of degree 1 in H.

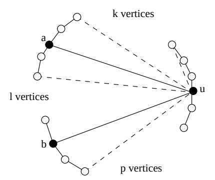
The base case is $n = 12$; suppose for the sake of contradiction that we can add 8 edges to C_{12} to obtain a graph of diameter 2. If t is the number of components of H, then as there are at least $12 - t$ edges in H, we have $t \geq 4$. Since we are not in any previous case, each component must have a vertex of degree greater than 1. Thus, H has 4 components each of which has one vertex of degree 2 and two vertices of degree 1. However, since we are not in Case 3, no two leaves can be connected in C_{12}, which is impossible.

Now suppose that we have proven the theorem for $k \leq n - 1$. If in $C_n \cup H$ there exists a triangle formed with one of the edges of C_n then $e(H) \geq n - 3$, since we can contract the edge of the triangle that is also an edge of C_n, and obtain a graph with $n - 1$ vertices whose diameter is still 2.

As previously noted, we have at least eight vertices $v_1, ..., v_8$ of degree 1, no two connected in H or C_n; the latter condition necessitates $n \geq 16$. In order to have $d(v_i, v_j) \leq 2$ all these eight vertices must be adjacent to a common vertex u, and thus $deg_H(u) \geq 6$.

Let a and b be two vertices of degree 1 in H at distance at least three in C_n connected with u, and let a_1, a_2, b_1, b_2 be adjacent to a and b respectively in C_n. By reasoning similar to that used in the cases above we have $deg_H(a_1) + deg_H(a_2) + deg_H(u) \geq n - 7$ and $deg_H(b_1) + deg_H(b_2) + deg_H(u) \geq n - 7$.

We want to calculate the maximum degree of u in H. This maximum occurs when there is an edge from u to every other vertex different than a and b and such that there are no triangles formed with an edge of C_n. Let k, l, p be the number of vertices between u and a, a and b, b and u respectively. Then



the maximum numbers of edges in H between these vertices and u are respectively: $\frac{k-3+1}{2}$, $\frac{l-2+1}{2}$, $\frac{p-3+1}{2}$. Note that $k + l + p + 3 = n$. As each of the edges counted above contributes one to the degree of u in H and also considering the edges ua, and ub, we obtain that $deg_H(u) \leq 2 + \frac{k+l+p-5}{2} = 2 + \frac{n-8}{2} = \frac{n-4}{2}$. So, $deg_H(u) \leq \frac{n-4}{2}$. Adding the inequalities above involving the degrees of a_1, a_2, u, b_1 and b_2 in H and using the last inequality we obtain: $n - 4 + deg_H(a_1) + deg_H(a_2) + deg_H(b_1) + deg_H(b_2) \geq 2n - 14$. Thus, $deg_H(a_1) + deg_H(a_2) + deg_H(b_1) + deg_H(b_2) \geq n - 10$. Remembering that there are at least 6 edges incident to u not counted yet in the last inequality, we obtain $e(H) \geq deg_H(u) + deg_H(a_1) + deg_H(a_2) + deg_H(b_1) + deg_H(b_2) \geq n - 4$.

Finally, let w any vertex of degree greater than 1 in H, not already counted as a neighbor of u in C_n, and v another vertex of degree 1 in H, not a neighbor of a_1, a_2, b_1, b_2 or u. In order to get from w to v on a 2-length path of $H \cup C_n$ there must exist the edge uw or one edge from w to one of v's neighbors. In both cases there must be at least one edge more. This ends the proof.

□

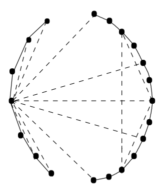
3. Diameter 3

We now present some results for diameter 3.

Theorem 3. *(Alon, Gyárfás, Ruszinkó)*
Suppose that G is a graph of order n with maximum degree $D \geq 2$. Then at least $n-3(D+1)^3-2(D+1)^2-1$ edges are needed to extend G into a graph of diameter three.

Corollary 4. *(Alon, Gyárfás, Ruszinkó)*
At least $n - 100$ edges must be added to C_n to get a graph of diameter three.

Alon et al. conjectured that the minimum number of edges needed to be added is actually $n - 6$ by showing a specific construction. However, the following counterexample shows that at most $n - 8$ edges need to be added.



Using the argument of Alon et al., we improve their lower bound as follows.

Theorem 5. *At least $n - 59$ edges must be added to C_n in order to obtain diameter 3.*

Proof. As before, let H be the graph consisting of the vertices of C_n and the edges added to C_n to decrease the diameter to 3. If t is the number of the tree components of H, as in [1] define the **fixed edges** to be the edges of the trees and the edges of a unicyclic spanning subgraph for each of the other components; note that there are $n-t$ fixed edges. Let $x_1, \ldots, x_t$ denote vertices of degree 1 in the tree components, and again as in [1] define an **essential path** to be a path of length 3 between x_i and x_j which has its middle edge in H.

Observe that as the diameter is 3 there must be a path of length at most 3 between all pairs of vertices $\{x_i, x_j\}$. From a vertex x_i at most 6 vertices x_j can be reached by a non-essential path starting with an edge in G. Also, from x_i at most 4 other x_j's can be reached by non-essential paths starting with an edge in H. Thus, at most 10 vertices x_j can be reached from x_i by nonessential paths, so there are at most $\frac{10t}{2} = 5t$ nonessential paths from x_i to x_j.

One middle edge in an essential path can be contained in at most 6 essential paths. Indeed, any such path has 2 edges of C_n at its ends and at most one more edge of H that ends in an x_k.

The number of vertices spanning the fixed middle edges of the essential paths is $3t$ as they are adjacent to an x_i, and there are at most 3 edges in $C_n \cup H$ from every x_i. Then, the number of middle edges in the essential paths is at most the number of vertices spanning them.

The considerations above give the following lower bound for the non-fixed edges of H: $\frac{\binom{t}{2}-5t}{6} - 3t = \frac{t(t-1)}{12} - \frac{23}{6}t$. So, if $t \geq 59$ then $e(H) \geq n - t + \frac{t(t-1)}{12} - \frac{23}{6} \geq n - 59$. Otherwise, $e(H) > n - t > n - 59$. □

We end the paper by stating the following conjecture suggested by verifications for small cases.

Conjecture 1. *For $n \geq 12$, at least $n-8$ edges have to be added to C_n in order to obtain a graph of diameter 3.*

Acknowledgements

This work was done under the supervision of Joseph Gallian at the University of Minnesota Duluth, with financial support from Bard College. The author wishes to thank Stephen Hartke, Mike Develin, Matt Riddle and Joseph Gallian for many useful suggestions.

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[1] Noga Alon, András Gyárfás, and Miklós Ruszinkó, Decreasing the diameter of bounded degree graphs, *Journal of Graph Theory*, Vol.35(3) (2000), 161-172.

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