Lecture 8

- The Fourier analytic method
- Testing linearity / testing membership in the Hadamard code

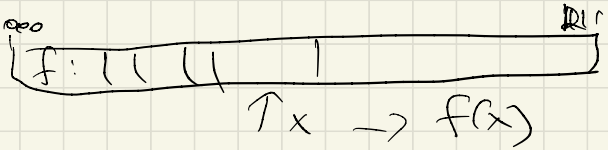
$\tilde{F} \subseteq \{ f : \mathbb{F}_2^n \to \mathbb{F}_2 \}$ is a collection of Boolean fncs.

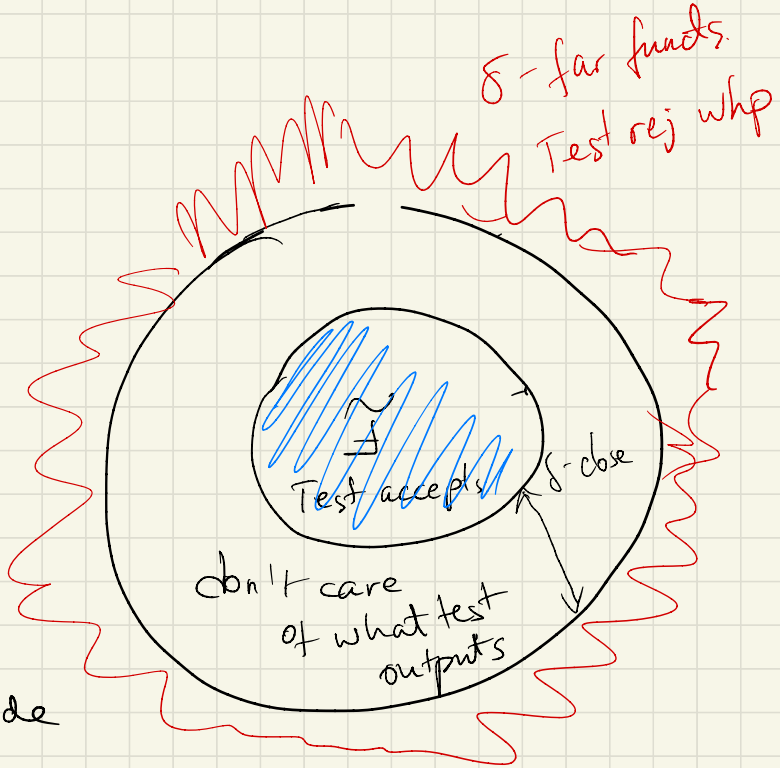
$\tilde{F}$ is $(k, \varepsilon_1, \varepsilon_2, \delta)$-locally testable if $\exists$ rand alg A s.t., given oracle access to input $f : \mathbb{F}_2^n \to \mathbb{F}_2$

- A makes k queries $x_1, \dots, x_k$
- Based on $f(x_1), f(x_2) \dots f(x_k)$
  - if $f \in \tilde{F}$ acc wp $1 - \varepsilon_1$
  - if f is δ-far from $\tilde{F}$ rej wp $\geq \varepsilon_2$

if $\varepsilon_1 = 0$ 1-sided.

$$d(f,g) = \frac{\#\, x \text{ s.t. } f(x) \neq g(x)}{2^n}$$ (relative distance)





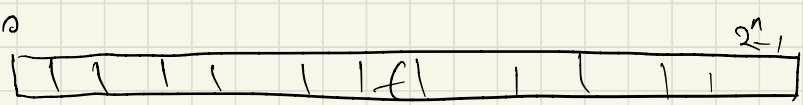
$d(f, \mathcal{F}) = \min_{g \in \mathcal{F}} d(f, g)$

Hadamard code

$$Lin = \{ f_a : \mathbb{F}_2^n \to \mathbb{F}_2 \mid f_a(x) = \langle a, x \rangle \}$$

$a \in \mathbb{F}_2^n$

$\langle \cdot, \cdot \rangle$ ← inner prod mod 2.



Q: is f lin or δ-far from linear.

Loc. view of Lin: $f_a(x) + f_a(y) = f_a(x+y)$

$\forall f_a \in Lin$

x, y, $x+y$ ← queries.

Test: pick x, y unif at rand from $\mathbb{F}_2^n$

If $f(x) + f(y) = f(x+y)$ acc

else rej.

Analysis: if $f \in Lin$ Test acc.

if δ-far ?

Q: what is $\Pr[\text{rej} \mid \delta\text{-far func}]$

Fourier analytic method

$f_1 : \mathbb{F}_2^n \to \mathbb{F}_2$ $\xrightarrow{\text{embed into } \mathbb{R}}$ $f(x) = (-1)^{f(x)}$

$b \to (-1)^b$

$0 \to 1$

$1 \to -1$

$f_a : \mathbb{F}_2^n \to \mathbb{F}_2$

$f_a \to \chi_a : \mathbb{F}_2^n \to \{\pm 1\}$.

Useful props:

- $$\mathop{\mathbb{E}}_{x \in \mathbb{F}_2^n} \chi_a(x) = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{ow.} \end{cases}$$

$e_i = 00 \cdot 1000$

Pf: $\Downarrow$

$$\frac{1}{2^n} \sum (-1)^{a \cdot x}$$

$\downarrow$

- if $a = 0$ $\quad 1$
- if $a \neq 0$ $\quad a_i = 1$ for $i \in [n]$.

$$\frac{1}{2}\left(\sum_{\substack{x_i = 0 \\ x}} (-1)^{a \cdot x} + \sum_{\substack{x_i = 1 \\ x}} (-1)^{a \cdot x} \right)$$

$$= \frac{1}{2}\left(\sum_{x_i = 0} \left((-1)^{a \cdot x} + (-1)^{a x + a \cdot e_i} \right) \right)$$

$$= \frac{1}{2}\left(\sum_{x_i = 0} (-1)^{a \cdot x} - (-1)^{a \cdot x} \right.$$

$a \cdot e_i = a_i = 1$

Multiplc. of charac.

- $\forall\, a, x, y \in \mathbb{F}_2^n$ $\quad \chi_a(x)\cdot\chi_a(y) = \chi_a(x+y)$

$$(-1)^{a\cdot x}\ (-1)^{a\cdot y} = (-1)^{a\cdot(x+y)}$$

- $\forall\, a, b, x$ $\quad \chi_a(x)\ \chi_b(x) = \chi_{a+b}(x)$

Inner prod:

$$f, g : \mathbb{F}_2^n \to \mathbb{R}$$

$$\langle f, g\rangle \overset{\text{def}}{=} \frac{1}{2^n} \sum_{x\in\mathbb{F}_2^n} f(x)\, g(x) = \mathop{\mathbb{E}}_x\left[f g(x)\right]$$

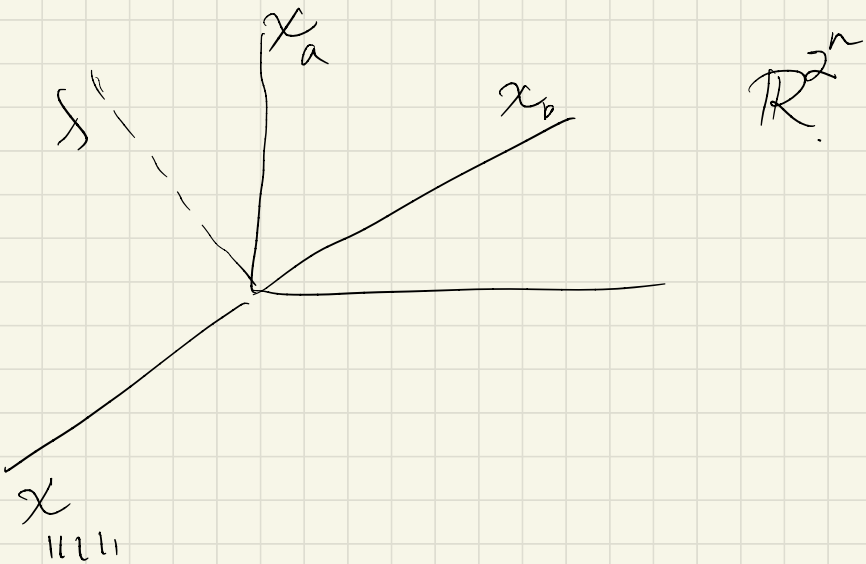
Prop: $\{\chi_a\}_{a\in\mathbb{F}_2^n}$ forms an orthonormal basis of $\mathbb{R}^{2^n}$

$a+b=0$
$a=b$.

Pf, $\langle \chi_a, \chi_a\rangle = \frac{1}{2^n}\sum_x \chi_a^2(x) = 1$

$a\neq b$ $\quad \langle \chi_a, \chi_b\rangle = \frac{1}{2^n}\sum \chi_a(x)\,\chi_b(x) =$

$$= \frac{1}{2^n}\sum_x \chi_{a+b}(x) = \mathbb{E}\,\chi_{a+b}(x) = 0$$



Can write $f: \mathbb{F}_2^n \to \mathbb{R}$ as

$$f(x) = \sum_a \hat{f}_a \chi_a(x)$$

$\hat{f}_a$: Fourier coefficients

f: Fourier expansion of f.

Prop: $\hat{f}_a = \langle f, \chi_a \rangle$

Pf: $\langle f, \chi_a \rangle = \left\langle \sum_b \hat{f}_b \chi_b(x), \chi_a(x) \right\rangle$

$$= \frac{1}{2^n} \sum_x \sum_b \hat{f}_b \chi_b(x) \chi_a(x)$$

$$= \sum_b \hat{f}(b) \, \mathbb{E}_x[\chi_{b+a}(x)]$$

$$= \hat{f}_{b=a} \qquad \uparrow \quad \begin{cases} 0 & \text{if } b+a \neq 0 \\ 1 & b+a=0 \\ & b=a \end{cases}$$

$$= \hat{f}_a$$

Parseval's.

$\forall$ Bool $f : \mathbb{F}_2^n \to \{\pm 1\}$

$$\langle f, f \rangle = \mathbb{E}_x f^2(x) = \sum_{a \in \mathbb{F}_2^n} \hat{f}_a^2 = 1$$

Pf: $\langle f, f \rangle = \langle \sum \hat{f}_a \chi_a(x), \sum \hat{f}_b \chi_b(x) \rangle$

Prop: $\langle f, g \rangle = 1 - 2d(f,g)$

$f, g : \mathbb{F}_2^n \to \{\pm 1\}$.

$d(f,g) = d(f, g+1)$

etc.

Cor: $\langle f, \chi_a \rangle = \widehat{f}_a = 1 - 2d(f, \chi_a)$

$\chi_a(x) = (-1)^{a \cdot x} \longrightarrow \text{Lin} = \{\chi_a\}_{a \in \mathbb{F}_2^n}$

Cor: $d(f, \{\chi_a\}_{\text{Lin}}) = \frac{1}{2} - \frac{1}{2} \max_{a \in \mathbb{F}_2^n} \widehat{f}_a$

Test linearity: Pick x, y

T^f: $f(x) f(y) = f(x+y)$

$\{\pm 1\}$ $\{\pm 1\}$ $\{\pm 1\}$.

Lemma: If T^f rej wp $\leq \delta$ then

$d(f, \text{Lin}) \leq \delta$.

Counterpositive of:

If f is δ-far then T^f rej wp δ.

Pf: $I_{\alpha,\beta} = \begin{cases} 0 & \text{if } f(\alpha)f(\beta) = f(\alpha+\beta) \\ 1 & \text{ow} \end{cases}$

$= \frac{1}{2} - \frac{1}{2}(f(\alpha)f(\beta)f(\alpha+\beta))$

if $f(\alpha)f(\beta) = f(\alpha+\beta)$

$= \frac{1}{2} - \frac{1}{2} = 0$

ow: $\frac{1}{2} + \frac{1}{2} = 1$

$\Pr[T^f \text{ rej}] = \mathbb{E}[I_{\alpha,\beta}] = \frac{1}{2} - \frac{1}{2} \mathop{\mathbb{E}}_{\alpha,\beta \in \mathbb{F}_2^n} f(\alpha)f(\beta)f(\alpha+\beta)$

Claim: $\mathop{\mathbb{E}}_{\alpha,\beta} f(\alpha)f(\beta)f(\alpha+\beta) = \sum_{a \in \mathbb{F}_2^n} \hat{f}_a^3$

Pf: $\mathop{\mathbb{E}}_{\alpha,\beta} \left(\sum_a \hat{f}_a \chi_a(\alpha)\right)\left(\sum_b \hat{f}_b \chi_b(\beta)\right)\left(\sum_c \hat{f}_c \chi_c(\alpha+\beta)\right)$

$= \mathop{\mathbb{E}}_{\alpha\beta} \sum_{abc} \hat{f}_a \hat{f}_b \hat{f}_c \chi_a(\alpha)\chi_b(\beta)\chi_c(\alpha+\beta)$

$= \chi_a(\alpha)\chi_c(\beta)$

$$= \sum_{a,b,c} \hat{f}_a \hat{f}_b \hat{f}_c \; \mathbb{E}_{\alpha,\beta} \chi_{a+c}(\alpha)\chi_{b+c}(\beta)$$

$$= \sum_{a,b,c} \hat{f}_a \hat{f}_b \hat{f}_c \left(\mathbb{E}_{\alpha} \chi_{a+c}(\alpha)\right)\left(\mathbb{E}_{\beta} \chi_{b+c}(\beta)\right)$$

$\mathbb{E}_{\alpha} \chi_{a+c}(\alpha)$: $0,1$; $\mathbb{E}_{\beta} \chi_{b+c}(\beta)$: $0,1$

only survives

$$\left.\begin{array}{l} a+c=0 \\ b+c=0 \end{array}\right\} \Rightarrow a=b=c$$

$$= \sum_{a} \hat{f}_a^3 \qquad \checkmark$$

Hence $\Pr[\text{Test rejt}] = \frac{1}{2} - \frac{1}{2}\sum \hat{f}_a^3$.

WTS: if $\Pr[\text{Test rej}] < \delta$ Then $d(f, \text{Lin}) < \delta$

$d(f, \text{Lin}) = 1/2 - 1/2 \max_a \hat{f}_a$ WTS $< \delta$

Enough to show $d(f,\text{Lin}) < \Pr[\text{Test rej}]$

Then $d(f,\text{Lin}) < \delta$ ✓.

Claim: $d(f, \text{Lin}) \leq \Pr[\text{Test rej}]$.

$$\frac{1}{2} - \frac{1}{2}\max_a \hat{f}_a \overset{\text{WTS}}{\leq} \frac{1}{2} - \frac{1}{2}\sum_a \hat{f}_a^3$$

equiv. $\sum_a \hat{f}_a^3 \leq \max_a \hat{f}_a$

Pf: $\sum_a \hat{f}_a^3 \leq \sum_a \hat{f}_a^2 \cdot (\max_b \hat{f}_b)$

$$\leq \left(\sum_a \hat{f}_a^2\right)(\max_b \hat{f}_b)$$

$$= 1 \cdot (\max_b \hat{f}_b)$$

Def. Dictator: $f(x) = x_i$ for some i.

$\chi_{001000} = \chi_{\{i\}} = x_i$

$= (-1)^{x_i}$

Subset of lin functions

Hastad's test: Given δ, f,

pick x, y, w

w: $\forall i$ (independently) $\Pr[w_i = 1] = \delta$

$w \leftarrow$ "δ-biased distribution"

Test: acc iff $f(x)f(y)f(x+y+w) = 1$

Setup

$$I_{x,y,w} = \frac{1}{2} - \frac{1}{2} f(x)f(y)f(x+y+w)$$

↳ indic

$$\Pr[\text{Hastad's test rejects}] = \mathop{\mathbb{E}}_{\substack{x,y \\ w \leftarrow \delta\text{ biased}}}[I_{x,y,w}]$$

Can show that:

Thm: Given $f: \{\pm 1\}^n \to \{\pm 1\}$.

$$\Pr[\text{Hastad's test rej}] = \frac{1}{2} - \frac{1}{2}(1-2\delta)^{|a|} \hat{f}_a$$

$|a| = |\text{supp } a|$

eg: $a = 11000$ $|a| = 2$